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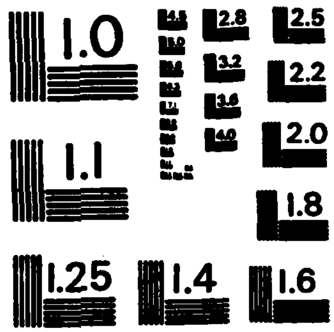

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DYNAMICAL DETERMINATION OF THE SCALES OF TURBULENCE

AERONAUTICAL RESEARCH ASSOCIATES OF PRINCETON, INC.
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APRIL 1982

FINAL REPORT
ON
INVARIANT TWO-POINT CORRELATION MODEL
OF TURBULENT FLOWS

FOR PERIOD 1 MARCH 1981 - 30 APRIL 1982

PREPARED FOR
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Integral scales for fully developed turbulent flows may be defined as moments of the two-point tensor correlations among the dynamical variables which characterize the flow, for example velocity and temperature. We show in this paper that this concept of integral scales allows for a rigorous derivation of rate equations for the integral scales (at a single point) if two major assumptions are made. The first assumption is that the integral scales are sufficiently small (compared with the scale of spatial variations of the mean quantities) to enable one to neglect the difference

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between convection at two points separated by less than the integral scale itself. This assumption is necessary to obtain convections of the integral scale (which is a one-point function) at a single point and thus represents the most general physical requirement for the very existence of closed equations for the integral scales. The assumption which is mathematically embodied by the moment expansion of the correlation tensor (Section 1) restricts the rigorous theory to nearly homogeneous flows. We shall see (Section 2) that our theory is in satisfactory agreement with the experimental evidence for nearly homogeneous flows. We follow convention in making the bold suggestion that the theory be applied outside the rigorous limits of its derivation for at least the reason that at the present there seems to be no sound theoretical alternative to solving for the full two-point tensors in detail when near-homogeneity fails.

The second major assumption that we find necessary for a rigorous derivation of rate equations for the integral scales stems from a coupling between the evolution of the Reynolds stress and other one-point correlations to the evolution of the integral scale. The nature of this coupling is, in our opinion, only partially understood at present. To proceed, we observe that we can obtain from the two-point correlations at every space point and at every instant of time, both the Reynolds stress (by collapsing the two points to one), and the integral scale (by a suitable integration in relative separation).

In view of this fact, it appears natural to propose that the concepts of second-order closure, which have been successfully developed over the past decade, be extended from applying to one-point correlations to apply generally to the two-point correlations. It is understood that the "simplest" proven equations are to be considered for the relevant models until these are proven unsatisfactory. We shall see in what follows that this assumption offers a valuable guide to obtaining rate equations for the integral scale and allows us to incorporate in the generalized theory all the major successes of currently available second-order closure.

It may be worthwhile to point out that we do not derive the scale equations from the Reynolds stress equations. Rather both equations follow from an equation for the two-point correlation tensor.

PREFACE

There is more than one way to look at any physical problem. In the field of turbulence research, it has become fashionable to define almost any unsteady flow in the wake of an unaccelerated body as turbulence. A lot of these unsteady flows contain large-scale, coherent structures and, thus, there has arisen a "large-scale eddy cult." The basis thesis of this group is that turbulence should or (for the extremists) must be attacked by some technique which identifies the turbulence with the interaction and decay of such large-scale structures. A corollary of this position is that closure methods do not or (for the extremists) cannot address themselves specifically to large-scale eddies and, therefore, are not really anything but dull, unphysical, and temporary methods for dealing with turbulent flows.

The authors of this paper do not believe that these people really understand the nature of closure calculations at the present time. Not only have closure methods demonstrated the existence of large-scale eddies in two cases (the roll eddies of the marine planetary boundary layer and unsteady large eddies in the flow behind a rearward-facing step (see references 11 and 12 in the main body of this report)), but these eddies have been resolved in all their gory detail. This can be accomplished when the closure equations are used in their elliptic, time-dependent form and the grid spacing is fine enough. Why is this so? It is because the Euler equations (which govern the formulation and a great deal of the behavior of large eddies) are contained in the time-dependent, elliptic equations. Thus we submit that closure techniques not only can describe large-scale eddies but must do so if the time-dependent, elliptic forms of the equations are used and the grid spacing is small enough to resolve these eddies.

One further point. The Karman vortex wake is an unsteady flow associated with an unaccelerated body. We do not prefer to think of it as a form of structured turbulence (nor did those who first studied the phenomenon) although, if one were a member of "the cult," this point of view might be taken. The reason that we choose not to consider it as structured turulence is that there is ample evidence of its existence in laminar flow.

From the closure point of view, the Karman wake is looked at as an unsteady flow peculiar to the body that produces it. This unsteady flow interacts with itself to decay either through laminar exchange and dissipation or through turbulent exchange and dissipation, depending on the Reynolds number.

The research reported here is a description of our first attempts to make a closure theory of turbulence that is compatible with the large-scale structures that we must inevitably find when we run our closure codes in an elliptic manner. The work reported is one completed step in this direction. As noted later in the text, we do not consider this work complete. We sincerely regret that we had to terminate this work before our attempt to construct a more general closure formulation could be completed. While we consider many of the detailed large-eddy studies to be outstanding, we also believe that elliptic, non-steady, closure techniques will be the backbone of turbulence computations that will be of use to the military for the next twenty years, and that closure techniques that are compatible with this approach should and will be pursued.

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1. INTRODUCTION

Thinking of the integral scales of turbulence as moments of the two-point correlation tensors generates a general approach to the dynamical determination of these scales. The purpose of this paper is to demonstrate the validity of this proposition and to develop a number of its consequences.

In order to calculate the level of turbulent kinetic energy, $q^2/2$, the separate levels in the "energy components," ($\overline{u'^2}$, $\overline{v'^2}$ and $\overline{w'^2}$), as well as the turbulent stresses (e.g., $\overline{u'v'}$), the second-order-closure approach has been developed to a high degree over the last decade. To make calculations of the turbulent fluctuations indicated above, it is necessary to provide information on the behavior of the turbulence scales that have been introduced in second-order models that represent turbulent transport, isotropization, and dissipation. It is desirable to obtain local rate equations for the turbulence scales because, for example, one of them represents the size of typical energy-containing eddies and this size varies considerably from point to point in a turbulent flow. A similar observation applies to the size of the eddies that are mainly responsible for the dissipation of the turbulence. We develop in this paper a technique to determine local rate equations for the integral as well as the microscales of turbulent flows. We follow and generalize the basic ideas introduced by G. I. Taylor[1] who focussed his attention on the important case of isotropic turbulence. The normalized autocorrelation function f(r) of the component of the velocity in the direction $\underline{r}$ with the same velocity component at a distance r has schematically the form indicated in Figure 1. Here f(r) is defined to be the normalized time (or space, or ensemble) average

$$f(r) = \frac{\left\langle u_r'(\underline{x})\ u_r'(\underline{x} + \underline{r})\right\rangle}{\left\langle u_r'^2(\underline{x})\right\rangle}$$

The typical curve is not a Gaussian except when the turbulence is very weak (e.g., in the final stages of decay of isotropic turbulence). It is therefore necessary to allow for two distinct spatial scales in order to

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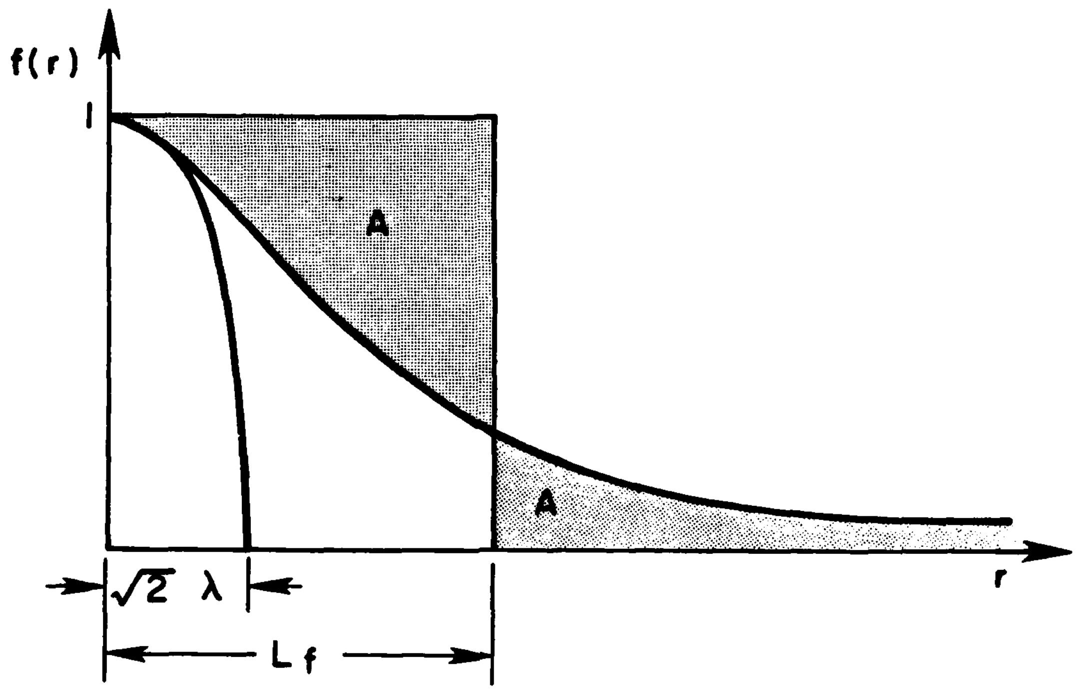

Figure 1. Geometrical interpretation of the two scales of turbulence, the dissipative scale, λ, and the average size of the energy containing eddies, L_f.

ly characterize turbulent flows, and this was indeed done by
ie smaller scale ("microscale") is defined by the curvature
:he origin

$$\lambda^2 = \lim_{r \to o} [-f''(r)]^{-1} \geq 0$$

shown to be characteristic of the eddies responsible for viscous
ı. The longer scale ("integral scale") is defined by Taylor in
as to account for the non-Gaussian tail of f as the area under
:, namely,

$$L_f = \int_0^\infty f(r)\, dr$$

ıles degenerate into a single scale as f becomes Gaussian. The
ı between the two scales can also be inferred from the form of the
ırum of the turbulent field because the power spectrum is
easily related to the Fourier transform of f(r). One geometric
the two scales λ and L_f is shown in Figure 1 where $\sqrt{2}\,\lambda$ is the
f the parabola tangent to f at the origin with the r axis, and L_f
ructed that the hatched areas (denoted by A) are equal. We shall
ylor's definitions yield, when appropriately generalized,
which transform as tensors and are therefore meaningful in flows
general geometry. We shall then show that the scales satisfy
d dynamical equations which we shall derive from the
es equations for fluid flow.

pose a generalized second-order closure which makes the equations
-point correlation functions self-contained. Thus, for the
elocity correlation tensor R_{ij}, we say that we have a
closure if we can write

$$\frac{\partial R_{ij}}{\partial t} = T_{ij}\,[R_{k\ell}]$$

where T is a tensor functional of R_{ij}. We shall restrict T considerably by requiring integrals to appear only through the integral scale tensor Λ_{ij} which we define, following Rotta[2], as a weighted moment of R_{ij}

$$\frac{q^2}{3}\,\Lambda_{ij}(\underline{x}_c) = \int \frac{dr}{4r\pi^2}\, R_{ij}\,(\underline{x}_c, \underline{r})$$

The two variables $\underline{x}_c$ and $\underline{r}$ are defined below. We observe that, for isotropic turbulence

$$\Lambda_{ij}^{(is)} = \delta_{ij}\,\frac{\Lambda_{kk}^{(is)}}{3}$$

The normalization factor has been adopted so that, for isotropic turbulence

$$\Lambda \equiv \frac{1}{3}\,\Lambda_{kk} = \frac{L_f + 2L_g}{3} = \frac{2}{3}\,L_f$$

It is interesting to note that this choice of normalization leads to a scalar scale Λ, which is 2/3 of the longitudinal integral scale and that the scale now defined is approximately equal to that generally used in current second-order-closure calculations. The basic reason for the definition adopted above for (generalized) second-order closure can now be made clear. The tensor R_{ij} contains the information needed to obtain the kinematic Reynolds stress

$$\overline{u_i' u_j'} = \lim_{\underline{y}\to\underline{x}} R_{ij}\,(\underline{x},\underline{y})$$

and also the scales that enter the modeled terms in the equation for $\overline{u_i' u_j'}$ if these are assumed to be related to the Λ_{ij} just introduced.

In order to derive local rate equations for Λ_{ij}, we find it necessary to generalize the standard integral methods to extract average information from equations in many independent variables. The procedure that we introduce is shown to coincide with the standard integral methods where these are applicable. Our generalization is made necessary in order to treat the two-point correlation tensors of the theory of turbulence because these are convected independently at two distinct points. The use of the moment expansion (which is necessary to obtain rate equations for scales at a single point) restricts the ensuing theory to nearly homogeneous flows. We show in the following that our theory is in satisfactory agreement with the experiments on nearly homogeneous flows. We believe that our theory (like the Chapman-Enskog theory of molecular transport) will give useful information outside the limits of its rigorous derivation.

1.1 Equations for the Two-Point Correlations

We consider incompressible flows with constant density which may carry a passive additive. We assume that the governing equations are the Navier-Stokes equations

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} = \nu \nabla_x^2 u_i \tag{1}$$

the continuity equation

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

and the convection-diffusion equation

$$\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} = D \nabla_x^2 \theta \tag{3}$$

The notation adopted is that of cartesian tensors with the following variables: the eulerian velocity is u_i, a field function of the independent variables $\underline{x} = (x_i)$ and t; the kinematic pressure is denoted by p which is the ratio of pressure to mass density; the passive additive is measured by θ which represents, for example, the temperature. The two transport coefficients are ν, the kinematic viscosity, and D, the diffusivity of the passive additive. The Laplacian operator at the point $\underline{x}$ is written as ∇_x^2 and it is defined by

$$\nabla_x^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \tag{4}$$

We introduce the Reynolds decomposition of the dependent variables by writing

$$u_i = \overline{u_i} + u_i' \tag{5}$$

$$p = \overline{p} + p' \tag{6}$$

$$\theta = \overline{\theta} + \theta' \tag{7}$$

where the average can be understood as an ensemble average or as a space (or time) average. In this latter case it is assumed that there is a marked separation between the short (or small) scale of the turbulent fluctuations and the large (long) scale of the variation of mean quantities. The averages are thereby always interchangeable with the space and time derivatives. The transport coefficients are assumed constant. For any arbitrary quantity A, we shall use interchangeably the overbar or the bracket to denote the turbulent average. Thus

$$\overline{A} \equiv \langle A \rangle \tag{8}$$

We now introduce the following two-point correlation tensors

$$R(\underline{x}, \underline{y}) = \langle \theta'(\underline{x})\, \theta'(\underline{y}) \rangle \equiv R \tag{9}$$

$$R_i(\underline{x}, \underline{y}) = \langle u_i'(\underline{x})\ \theta'(\underline{y})\rangle \equiv R_i \tag{10}$$

$$R_{ij}\ (\underline{x}, \underline{y}) = \langle u_i'(\underline{x})\ u_j'(\underline{y})\rangle \equiv R_{ij} \tag{11}$$

The order of indices and arguments is essential because the correlation tensors are not symmetric under separate exchange of indices and of positions. Note that when differentiating the argument can be omitted without ambiguity when a function of a single argument is differentiated. Note also that the Reynolds stress $\langle u_i'u_j'\rangle$ is the limit of R_{ij} as $\underline{y} \to \underline{x}$.

Substituting the Reynolds decompositions, Eqs. (5) - (7), into the basic Eqs. (1) - (3) we obtain, for the momentum equation

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial u_i'}{\partial t} + \bar{u}_k \frac{\partial \bar{u}_i}{\partial x_k} + \bar{u}_k \frac{\partial u_i'}{\partial x_k}$$

$$+ u_k' \frac{\partial \bar{u}_i}{\partial x_k} + u_k' \frac{\partial u_i'}{\partial x_k} + \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial p'}{\partial x_i} = \nu\, \nabla_x^2\, \bar{u}_i + \nu\, \nabla_x^2\, u_i' \tag{12}$$

for the continuity equation:

$$\frac{\partial \bar{u}_i}{\partial x_i} + \frac{\partial u_i'}{\partial x_i} = 0 \tag{13}$$

and for the convection-diffusion equation:

$$\frac{\partial \bar{\theta}}{\partial t} + \frac{\partial \theta'}{\partial t} + \bar{u}_k \frac{\partial \bar{\theta}}{\partial x_k} + \bar{u}_k \frac{\partial \theta'}{\partial x_k} + u_k' \frac{\partial \bar{\theta}}{\partial x_k} + u_k' \frac{\partial \theta'}{\partial x_k} = D\nabla_x^2\, \bar{\theta} + D\, \nabla_x^2\, \theta' \tag{14}$$

We average the "total" Eqs. (12) through (14) and remember that the averages are so defined as to commute with the space and time derivative. The result in "mean convection" form is, for momentum

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_k \frac{\partial \overline{u}_i}{\partial x_k} + \frac{\partial \overline{p}}{\partial x_i} = - \frac{\partial}{\partial x_k} \langle u_u' u_i' \rangle + \nu \nabla_x^2 \overline{u}_i \tag{15}$$

For the continuity condition we have

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{16}$$

which was used to give the form exhibited for the right-hand side of Eq. (15). For the passive additive we find

$$\frac{\partial \overline{\theta}}{\partial t} + \overline{u}_k \frac{\partial \overline{\theta}}{\partial x_k} = - \frac{\partial}{\partial x_k} \langle u_k' \ \theta' \rangle + D \nabla_x^2 \overline{\theta} \tag{17}$$

We remember Reynold's basic observation that turbulence gives effective transport effects to the fluid in the form of a kinematic-stress tensor, $\langle u_u' \ u_i' \rangle$, and of a flux vector, $\langle u_u' \ \theta' \rangle$, for the passive additive.

Subtracting the mean from the total equations, we obtain the starting equations for the determination of the rate equations of the correlation tensors given by Eqs. (9) through (11). These equations are the fluctuation equations:

for momentum

$$\frac{\partial u_i'}{\partial t} + \overline{u}_k \frac{\partial u_i'}{\partial x_k} + \frac{\partial p'}{\partial x_i} = - u_k' \frac{\partial \overline{u}_i}{\partial x_k} - \frac{\partial}{\partial x_k} u_k' u_i' - \langle u_k' u_i' \rangle + \nu \nabla_x^2 u_i' \tag{18}$$

for continuity

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{19}$$

and for the fluctuations of the passive additive

$$\frac{\partial\theta'}{\partial t} + \bar{u}_k \frac{\partial\theta'}{\partial x_k} = -u_k' \frac{\partial\theta'}{\partial x_k} - \frac{\partial}{\partial x_k}\left[u_k'\theta' - \langle u_k'\theta'\rangle\right] + D\,\nabla_x^2\theta' \tag{20}$$

The fluctuation equations have been put in the "mean convection" form in order to emphasize that, for the fluctuations, the governing equations are of the primitive form with <u>two</u> modifications: turbulent transport tensors, $[u_u' u_i' - \langle u_u' u_i'\rangle]$ and $[u_u' \theta' - \langle u_u' \theta'\rangle]$, respectively and also turbulent "production" $-u_u' (\partial\bar{u}_i/\partial_{xu})$ and $-u_u' (\partial\bar{\theta}/\partial x_u)$, respectively.

We can now derive rate <u>and</u> continuity equations for R_{ij}. Multiply the $\partial u_i'(\underline{x})/\partial t$, Eq. (18), by $u_j'(\underline{y})$ and the equation for $\partial u_j'(\underline{y})/\partial t$ (derived from Eq. 18) by $u_i'(\underline{x})$. Then add and average. Using the definition, Eq. (11), we obtain

$$\begin{aligned}\frac{\partial R_{ij}}{\partial t} &+ \left[\bar{u}_k(\underline{x}) \frac{\partial}{\partial x_k} + \bar{u}_k(\underline{y}) \frac{\partial}{\partial y_k}\right] R_{ij} + \left\langle \frac{\partial p'}{\partial x_i} u_j'(\underline{y}) + u_i'(\underline{x}) \frac{\partial p'}{\partial y_j} \right\rangle \\ &= -\left[R_{ik} \frac{\partial \bar{u}_j}{\partial y_k} + \frac{\partial \bar{u}_i}{\partial x_k} R_{ij}\right] - \left[\frac{\partial}{\partial x_k} \langle u_k'(\underline{x}) u_i'(\underline{x}) u_j'(\underline{y})\rangle \right. \\ &\quad \left. + \frac{\partial}{\partial y_k} \langle u_k'(\underline{y}) u_i'(\underline{x}) u_j'(\underline{y})\rangle \right] + \nu(\nabla_x^2 + \nabla_y^2) R_{ij}\end{aligned} \tag{21}$$

We observe that the index i systematically accompanies the argument $\underline{x}$ and that the index j systematically accompanies the argument $\underline{y}$. The terms that require modeling are those for which R_{ij} does not appear explicitly. They are the pressure velocity correlation and the triple velocity correlation. It is of interest to note a first important difference between the familiar equation for the Reynolds stress tensor $\langle u_i' u_j'\rangle$ and Eq. (21) for the two-point correlation tensor R_{ij}. In the equation for the Reynolds stress, the viscous term requires closure since this is given by

$$\left(\frac{\partial}{\partial t} \langle u_i' u_j'\rangle\right)_{visc} = \lim_{\underline{y}\to\underline{x}} \left[\nu\,(\nabla_x^2 + \nabla_y^2)\, R_{ij}\,(\underline{x},\underline{y})\right] \tag{22}$$

which cannot be expressed, in general, in terms of $\overline{u_i' u_j'}$ alone. By contrast, the viscous term in the equation for R_{ij} does not require modeling.

In addition to the momentum equation, Eq. (21), R_{ij} also satisfies two continuity equations. Thus multiplying Eq. (19) by $u'_j(\underline{y})$ we have

$$u'_j(\underline{y})\,\frac{\partial u'_i(\underline{x})}{\partial x_i} = 0 \qquad (23)$$

which, when averaged, reads

$$\left\langle u'_j(\underline{y})\,\frac{\partial u'_i(\underline{x})}{\partial x_i}\right\rangle = 0 \qquad (24)$$

or

$$\frac{\partial}{\partial x_i} R_{ij}\,(\underline{x},\,\underline{y}) = 0 \qquad (25)$$

Similarly

$$\frac{\partial}{\partial y_j} R_{ij}\,(\underline{x},\,\underline{y}) = 0 \qquad (26)$$

Equations (25) and (26) constitute a second and, in our view, very important difference between the equation for the one-point Reynolds stress tensor and the two-point R_{ij}. In fact the stress $\overline{u'_i u'_j}$ does not satisfy a continuity equation as is well known from Lighthill's[3] approach to the problem of determining the noise generated by turbulence. By contrast, R_{ij} satisfies Eqs. (25) and (26) which will be seen to be stringent requirements on the modeling of the R_{ij} rate equation.

By a reasoning entirely parallel to the one outlined for the derivation of the rate equation for R_{ij}, we can obtain the rate equation for the flux vector R_i of the passive additive. The result of the calculation is

$$\frac{\partial\, R_i(\underline{x},\,\underline{y})}{\partial t} + \left[\overline{u}_k(\underline{x})\,\frac{\partial}{\partial x_k} + \overline{u}_k(\underline{y})\,\frac{\partial}{\partial y_k}\right] R_i(\underline{x},\,\underline{y}) + \left\langle\frac{\partial p'}{\partial x_i}\,\theta'(\underline{y})\right\rangle =$$

$$= - R_{ik} \frac{\partial \bar{\theta}}{\partial y_k} - R_k(\underline{x}, \underline{y}) \frac{\partial \bar{u}_i}{\partial x_k}$$

$$- \frac{\partial}{\partial x_k} \langle u_k'(\underline{x}) u_i'(\underline{x}) \; \theta'(\underline{y}) \rangle - \frac{\partial}{\partial y_k} \langle u_i'(\underline{x}) u_k'(\underline{y}) \; \theta'(\underline{y}) \rangle$$

$$+ (\nu \nabla_x^2 + D \nabla_y^2) R_i(\underline{x}, \underline{y}) \tag{27}$$

The reader can convince himself that no simple notational convention unambigously suppresses the arguments of the flux for the production term; hence the apparent pedantry in carrying the arguments of R_i explicitly. This result also applies to the continuity equation for the flux which reads

$$\frac{\partial}{\partial x_i} R_i(\underline{x}, \underline{y}) = 0 \tag{28}$$

We observed, when discussing R_{ij}, that the viscous term and the continuity conditions make for substantial differences between the equation for $\overline{u_i'u_j'}$ and for R_{ij}. The same concepts apply to Eqs. (27) and (28): (i) The viscous term in the equation for R_i is automatically closed and, (ii) the two-point flux equation must be compatible with the continuity requirements, Eq. (28). The physical meaning of the continuity requirements on R_{ij} and on R_i is best seen by differentiating Eq. (27) with respect to $\partial/\partial x_i$ and using Eq. (28). We obtain

$$0 + \frac{\partial}{\partial x_i} \left[\bar{u}_k(\underline{x}) \frac{\partial}{\partial x_k} R_i(\underline{x}, \underline{y}) \right] + 0 + \frac{\partial}{\partial x_i} \left\langle \frac{\partial p'}{\partial x_i} \theta'(\underline{y}) \right\rangle$$

$$= - \frac{\partial^2}{\partial x_i \partial x_k} \langle u_k'(\underline{x}) u_i'(\underline{x}) \theta'(\underline{y}) \rangle + 0 - \frac{\partial R_k(\underline{x},\underline{y})}{\partial x_i} \frac{\partial \bar{u}_i}{\partial x_k} \tag{29}$$

Equation (29) rearranges to

$$\nabla_x^2 \langle p'(\underline{x}) \theta'(\underline{y}) \rangle = -2 \frac{\partial \bar{u}_i}{\partial x_k} \frac{\partial R_k(\underline{x},\underline{y})}{\partial x_i} - \frac{\partial^2}{\partial x_i \partial x_k} \langle u_i'(\underline{x}) u_k'(\underline{x}) \theta'(\underline{y}) \rangle \tag{30}$$

which can be compared with the familiar Poisson equation for the pressure fluctuations

$$\nabla_x^2 p' = -2 \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial u'_j}{\partial x_i} - \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_j}{\partial x_i} - \frac{\partial^2 \langle u'_i u'_j \rangle}{\partial x_i \partial x_j} \tag{31}$$

obtained from the momentum equation for fluctuations, Eq. (18), by using the incompressibility of velocity fluctuation, Eq. (19). In fact, Eq. (31) multiplied by $\theta'(\underline{y})$ and averaged yields Eq. (30). An analogous reasoning applies to R_{ij}. In this case using Eq. (21) together with the continuity condition Eq. (25) gives

$$\nabla_x^2 \langle p'(\underline{x}) u'_k(\underline{y}) \rangle = -2 \frac{\partial \bar{u}_\alpha}{\partial x_\beta} \frac{\partial R_{\beta k}}{\partial x_\alpha} - \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \langle u'_\alpha(\underline{x}) u'_\beta(\underline{x}) u'_k(\underline{y}) \rangle \tag{32}$$

This equation can also be obtained by multiplying Eq. (31) by $u'_k(\underline{y})$ and averaging. We thus see that Eqs. (30) and (32) contain essential information on the long-range properties of the pressure fluctuations as given in detail by the Poisson equation (31). We believe that a model for R_{ij} that successfully incorporates Eqs. (30) and (31) would help clarify the structural as well as spectral properties of turbulent flows.

We complete this subsection by obtaining the rate equation for the temperature autocorrelation function $R = \langle \theta'(\underline{x}) \theta'(\underline{y}) \rangle$ as defined in Eq. (10). We multiply the equation for $\partial\theta'(\underline{x})/\partial t$ by $\theta'(\underline{y})$, add to it the equation for $\partial\theta'(\underline{y})/\partial t$ multiplied by $\theta'(\underline{x})$ and add. The result is the equation

$$\begin{aligned} \frac{\partial R}{\partial t} + \left[\bar{u}_k(\underline{x}) \frac{\partial}{\partial x_k} + \bar{u}_k(\underline{y}) \frac{\partial}{\partial y_k} \right] R = &- \left[R_k(\underline{x},\underline{y}) \frac{\partial \bar{\theta}}{\partial x_k} + R_k(\underline{y},\underline{x}) \frac{\partial \bar{\theta}}{\partial y_k} \right] \\ &- \frac{\partial}{\partial x_k} \langle u'_k(\underline{x})\ \theta'(\underline{x}) \theta'(\underline{y}) \rangle \\ &- \frac{\partial}{\partial y_k} \langle u'_k(\underline{y})\ \theta'(\underline{x})\ \theta'(\underline{y}) \rangle \\ &+ D\ (\nabla_x^2 + \nabla_y^2) R \end{aligned} \tag{33}$$

We ask patience for calling attention again to the arguments of the two-point flux vector. In parallel with R_{ij} and R_j, and by contrast with the equation for the one-point temperature variance, there is no need to model the viscous term in the two-point temperature correlation equation. No continuity requirement arises for R since the velocity does not enter its definition.

In summary, the basic equations for the correlation tensors and their continuity conditions are given by Eqs. (21), (25), and (26) for R_{ij}, Eqs. (27) and (28) for R_i and Eq. (33) for R.

We consider now the limit process that makes the two points of the correlation function collapse to a single point.

For this purpose we introduce a convenient coordinate system. The centroid vector $\underline{x}_c$ and the relative position vector $\underline{r}$ are defined by

$$\underline{x}_c = \frac{1}{2}(\underline{x} + \underline{y}) \tag{34}$$

$$\underline{r} = \underline{y} - \underline{x} \tag{35}$$

These equations are inverted by

$$\underline{x} = \underline{x}_c - \frac{1}{2}\underline{r}\ , \ \underline{y} = \underline{x}_c + \frac{1}{2}\underline{r} \tag{36}$$

The geometry of the transformation is shown in Figure 2. With these coordinates the collapse of the two distinct points $\underline{x}$ and $\underline{y}$ can be accomplished by taking the limit $\underline{r} \to 0$ holding $\underline{x}_c$ fixed; thus the two points $\underline{x}$ and $\underline{y}$ collapse to their centroid. To express the derivatives that appear in Eqs. (21), (27), and (33) for the correlation tensors and the continuity equations, (25), (26) and (28), we make use of the chain rules

$$\frac{\partial}{\partial x_i} = \frac{1}{2}\frac{\partial}{\partial x_{ci}} - \frac{\partial}{\partial r_i} \tag{37}$$

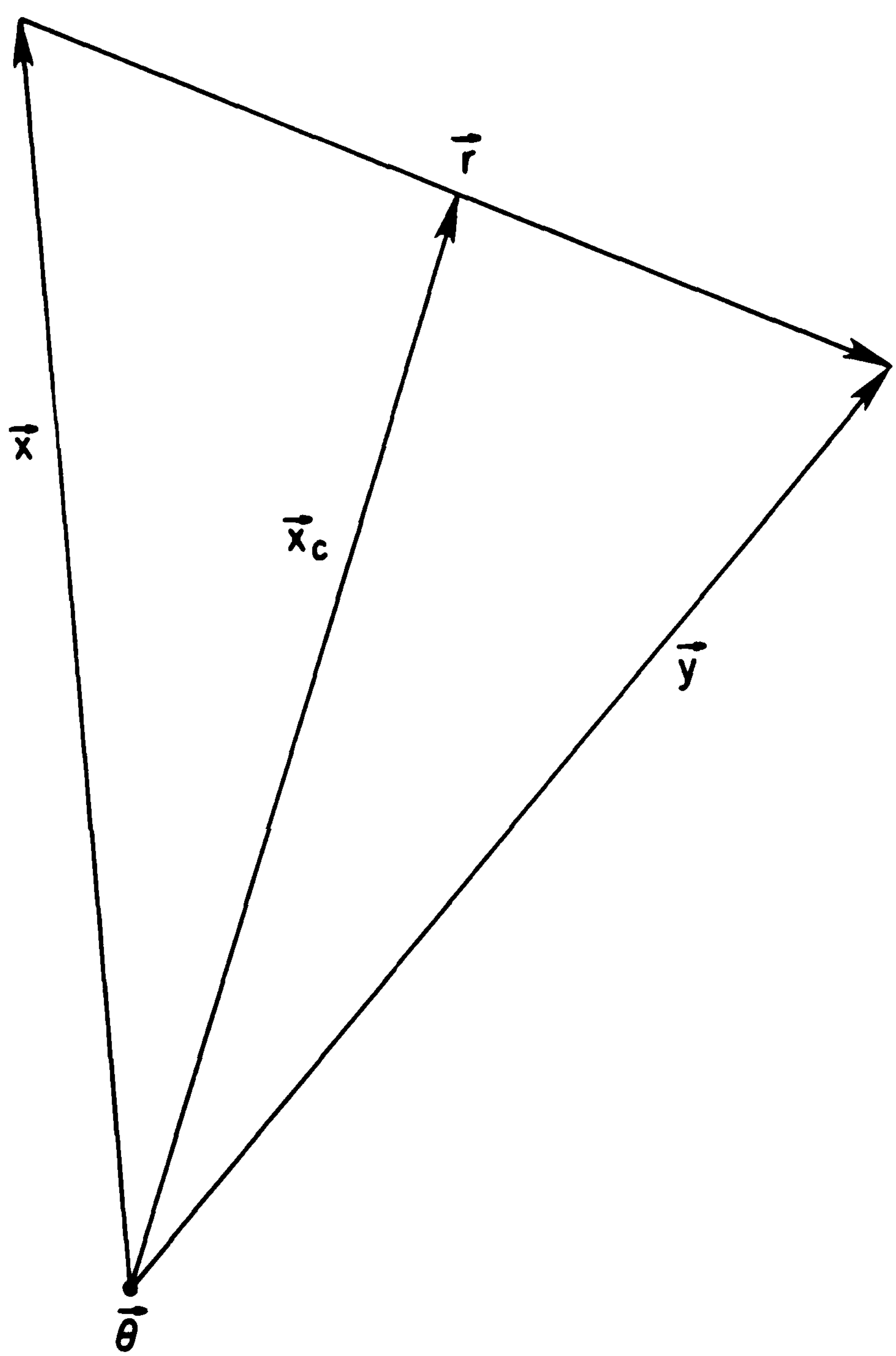

Figure 2. Geometry of centroid and relative position vectors.

$$\frac{\partial}{\partial y_i} = \frac{1}{2}\frac{\partial}{\partial x_{ci}} + \frac{\partial}{\partial r_i} \qquad (38)$$

We note that for functions, $F(\underline{x})$ of $\underline{x}$ only, i.e., functions that are independent of $\underline{y}$, we have

$$\frac{\partial F(\underline{x})}{\partial x_i} = \frac{\partial}{\partial x_{ci}} F\left(\underline{x}_c - \frac{1}{2}\underline{r}\right) \qquad (39)$$

Similarly, for functions $G(\underline{y})$, which are independent of $\underline{x}$, we have

$$\frac{\partial G(\underline{y})}{\partial y_i} = \frac{\partial}{\partial x_{ci}} G\left(\underline{x}_c + \frac{1}{2}\underline{r}\right) \qquad (40)$$

To illustrate the collapse process, we consider the divergence of the triple velocity correlation which has two terms in the R_{ij} equation and but a single term in the equation for $\langle u_i' u_j' \rangle$. We note first that the relation that holds in the collapse limit as defined above for the Reynolds tensor is

$$\langle u_i'(\underline{x}_c) u_j'(\underline{x}_c) \rangle = \lim_{\underline{r} \to 0} R_{ij}\left(\underline{x}_c - \frac{1}{2}\underline{r}, \underline{x}_c + \frac{1}{2}\underline{r}\right) \qquad (41)$$

Further, the turbulent kinetic energy is defined as $q^2/2$ where

$$q^2(\underline{x}_c) = \langle u_i'(\underline{x}_c) u_i'(\underline{x}_c) \rangle \qquad (42)$$

The two terms in Eq. (21) that contain triple velocity correlations are:

$$T_{ij}(\underline{x},\underline{y}) = \frac{\partial}{\partial x_k} \langle u_i'(\underline{x}) u_j'(\underline{y}) u_k'(\underline{x}) \rangle + \frac{\partial}{\partial y_k} \langle u_i'(\underline{x})\ u_j'(\underline{y})\ u_k'(\underline{y}) \rangle \qquad (43)$$

Introducing centroid and relative variables and using continuity of velocity fluctuations we have

$$T_{ij}\left(\underline{x}_c - \frac{1}{2}\underline{r}, \underline{x}_c + \frac{1}{2}\underline{r}\right) =$$

$$= \left\langle \left[\frac{\partial}{\partial x_{ck}} u_i'\left(\underline{x}_c - \frac{1}{2}\underline{r}\right)\right] u_k'\left(\underline{x}_c - \frac{1}{2}\underline{r}\right) u_j'\left(\underline{x}_c + \frac{1}{2}\underline{r}\right)\right\rangle$$

$$+ \left\langle u_i'\left(\underline{x}_c - \frac{1}{2}\underline{r}\right) u_k'\left(\underline{x}_c + \frac{1}{2}\underline{r}\right)\left[\frac{\partial}{\partial x_{ck}} u_j'\left(\underline{x}_c + \frac{1}{2}\underline{r}\right)\right]\right\rangle \quad (44)$$

We can now readily compute the limit as $\underline{r} \to 0$. Thus

$$\lim_{\underline{r}\to 0} T_{ij}\left(\underline{x}_c - \frac{1}{2}\underline{r}, \underline{r}_c + \frac{1}{2}\underline{r}\right)$$

$$= T_{ij}\ (\underline{x}_c, \underline{x}_c)$$

$$= \left\langle \left[\frac{\partial}{\partial x_{ck}} u_i'(\underline{x}_c)\right] u_k'\ (\underline{x}_c)\ u_j'(\underline{x}_c)\right\rangle + \left\langle u_i'(\underline{x}_c)\ u_k'(\underline{x}_c)\left[\frac{\partial}{\partial x_{ck}} u_j'(\underline{x}_c)\right]\right\rangle$$

$$= \frac{\partial}{\partial x_{ck}} \left\langle u_i'(\underline{x}_c)\ u_j'(\underline{x}_c)\ u_k'(\underline{x}_c)\right\rangle \quad (45)$$

Applying the limit $\underline{r} \to 0$ to Eq. (21), we find the well-known equation for the Reynolds stress tensor

$$\frac{\partial}{\partial t} \overline{u_i'u_j'} + \bar{u}_k \frac{\partial}{\partial x_{ck}} \overline{u_i'u_j'} + \left\langle \frac{\partial p'}{\partial x_{ci}} u_j' \right\rangle + \left\langle u_i' \frac{\partial p'}{\partial x_{cj}} \right\rangle$$

$$= - \left[\overline{u_i'u_u'} \frac{\partial \bar{u}_j}{\partial x_{cu}} + \frac{\partial \bar{u}_i}{\partial x_{ck}} \overline{u_k'u_j'}\right]$$

$$- \frac{\partial}{\partial x_{ck}} \overline{u_i'u_j'u_k'} + \nu\langle u_i'\ \nabla_c^2\ u_j'\rangle + \langle u_j'\ \nabla_c^2\ u_i'\rangle \quad (46)$$

where all correlations are evaluated at the centroid vector. Equation (45) is equivalent to the second term in the right-hand side of Eq. (46).

Application of the collapse process to the continuity equation, Eq. (25), gives

$$\frac{\partial}{\partial x_{ci}} \overline{u_i'u_j'} = 2 \lim_{\underline{x}\to 0}\left[\frac{\partial}{\partial r_i} R_{ij}\left(\underline{x}_c - \frac{1}{2}\underline{r}, \underline{x}_c + \frac{1}{2}\underline{r}\right)\right] \quad (47)$$

ns the fact that the Reynolds stress does not fulfill a simple ondition.

ce that the two identities commonly exploited to introduce the "tendency towards isotropy" and of "turbulent dissipation" have ues for the relevant two-point correlations.

pressure gradient velocity correlation in Eq. 45, we have the identity

$$\left\langle \frac{\partial p'}{\partial x_{ci}} u_j'(\underline{x}_c) \right\rangle = \frac{\partial}{\partial x_{ci}} \langle p' \; u_j' \rangle - \left\langle p' \frac{\partial u_j'}{\partial x_{ci}} \right\rangle \tag{48}$$

e obtain

$$+ \left\langle u_i' \frac{\partial p'}{\partial x_{cj}} \right\rangle$$

$$- \langle p' u_j' \rangle + \frac{\partial}{\partial x_{cj}} \langle p' u_i' \rangle \Big] \qquad \text{("pressure diffusion")}$$

$$\left\langle p' \left(\frac{\partial u_i'}{\partial x_{cj}} + \frac{\partial u_j'}{\partial x_{ci}} \right) \right\rangle \qquad \text{("tendency towards isotropy")} \tag{49}$$

t term is traceless as a consequence of continuity. For the pressure gradient-velocity correlations in Eq. (21) we can alogous decomposition, namely,

$$) \rangle + \left\langle u_i'(\underline{x}) \; \frac{\partial p'}{\partial y_j} \right\rangle$$

$$\frac{\partial}{\partial x_{ci}} \langle p'(\underline{x}) \; u_j'(\underline{y}) \rangle + \frac{\partial}{\partial x_{cj}} \langle u_i'(\overline{x}) p'(\underline{y}) \rangle \Big] \qquad \text{("two-point pressure diffusion")}$$

$$\left\langle p'(\underline{x}) \; \frac{\partial u_j'(\underline{y})}{\partial x_{ci}} \right\rangle + \left\langle p'(\underline{y}) \; \frac{\partial u_i'(\underline{x})}{\partial x_{cj}} \right\rangle \Big] \qquad \text{("two-point tendency toward isotropy")} \tag{50}$$

ation is justified by the fact that each of the two terms in square brackets in Eq. (50) reduces to the corresponding term in Eq. (49). We have, in fact, for the two-point tendency towards isotropy

$$T_{ij}^{(1)} \equiv \left\langle p'(\underline{x}) \frac{\partial u_j'(\underline{y})}{\partial x_{ci}} + p'(\underline{y}) \frac{\partial u_i'(\underline{x})}{\partial x_{cj}} \right\rangle$$

$$= \left\langle p'(\underline{x}_c - \frac{1}{2}\underline{r}) \frac{\partial}{\partial x_{ci}} u_j'\left(\underline{x}_c + \frac{1}{2}\underline{r}\right)\right\rangle$$

$$+ \left\langle p'\left(\underline{x}_c + \frac{1}{2}\underline{r}\right) \frac{\partial}{\partial x_{cj}} u_i'\left(\underline{x}_c - \frac{1}{2}\underline{r}\right)\right\rangle$$

In the limit as $r \to 0$ this becomes

$$\left\langle p'(\underline{x}_c) \frac{\partial u_j'(\underline{x}_c)}{x_{ci}} + p'(\underline{x}_c) \frac{\partial u_i'(\underline{x}_c)}{\partial x_{cj}} \right\rangle \tag{51}$$

Furthermore we note that the two-point tendency $T_{ij}^{(1)}(\underline{x},\underline{y})$ is trace free

$$T_{kk}^{(1)}(\underline{x}, \underline{y}) = 0 \tag{52}$$

The viscous term in the Reynolds stress equation is often rearranged using as an identity Leibintz's rule

$$\nabla_c^2 \, \overline{u_i' u_j'} = \langle u_i'(\nabla_c^2 u_j')\rangle + \langle(\nabla_c^2 u_c')u_j'\rangle + 2\left\langle \frac{\partial u_i'}{\partial x_{ck}} \frac{\partial u_k'}{\partial x_{ck}} \right\rangle \tag{53}$$

which can be used to rewrite the viscous term in Eq. (46) as

$$\nu \, \langle u_i' \, \nabla_c^2 u_j'\rangle + \langle u_j' \, \nabla_c^2 \, u_i'\rangle$$

$$= \nu\nabla_c^2 \, \langle u_i' u_j'\rangle \qquad \text{(viscous "diffusion" of the Reynolds stress)}$$

$$- 2\nu \left\langle \frac{\partial u_i'}{\partial x_{ck}} \frac{\partial u_j'}{\partial x_{ck}} \right\rangle \quad \text{("turbulent dissipation")} \tag{54}$$

In good analogy, we can write, for the two-point viscous term in Eq. (21):

$$\nu(\nabla_x^2 + \nabla_y)\, R_{ij} = \nu\langle u_i'(\underline{x})\ \nabla_y^2\, u_j'(\underline{y})\rangle + \nu\, \langle(\nabla_x^2 u_j'(\underline{x}))u_j'(\underline{y})\rangle$$

$$= \nu\ \nabla_c^2\ R_{ij} \qquad \text{(viscous "diffusion" of } R_{ij})$$

$$- 2\nu\left\langle \frac{\partial u_i'(\underline{x})}{\partial x_{ck}}\ \frac{\partial u_j'(\underline{y})}{\partial x_{ck}} \right\rangle \text{(two-point turbulent "dissipation")} \tag{55}$$

The identification is justified here as in the case of pressure-gradient-velocity correlations on the ground that we have term by term collapse of Eq. (55) to Eq. (54).

Before introducing the models adopted for our generalized second-order closure, we observe that the collapse of the two points in Eq. 27 for R_i gives the familiar rate equation for the one-point flux $\langle u_i'\theta'\rangle$

$$\frac{\partial}{\partial t}\langle u_i'\theta'\rangle + \overline{u}_k \frac{\partial}{\partial x_{ck}}\langle u_i'\theta'\rangle + \left\langle \frac{\partial p'}{\partial x_{ci}}\ \theta' \right\rangle = -\ \overline{u_i'u_k'}\ \frac{\partial \overline{\theta}}{\partial x_{ck}} - \overline{u_k'\theta'}\ \frac{\partial \overline{u}_i}{\partial x_{ck}}$$

$$- \frac{\partial}{\partial x_{ck}}\langle u_i'u_k'\ \theta'\rangle + \nu\ \langle\theta'\nabla_c^2 u_i'\rangle + D\ \langle u_i'\ \nabla_c^2\theta'\rangle \tag{56}$$

Similarly Eq. (33) for R becomes in the limit $\underline{r} \to 0$:

$$\frac{\partial}{\partial t}\overline{\theta'^2} + \overline{u}_k \frac{\partial}{\partial x_{ck}}\overline{\theta'^2}$$

$$= -\ 2\ \overline{u_k'}\ \overline{\theta}\ \frac{\partial\overline{\theta}}{\partial x_{ck}} - \frac{\partial}{\partial x_{ck}}\langle u_k'\ \theta'^2\rangle + D\ \nabla_c^2\ \overline{\theta'^2} - 2D\left\langle \frac{\partial\theta'}{\partial x_{ck}}\ \frac{\partial\theta'}{\partial x_{ck}} \right\rangle \tag{57}$$

We note that when the Prandtl number (ν/D) differs from unity, the molecular transport terms in the flux equation play a different role than those in the Reynolds stress and temperature variance equations.

1.2 Model Equations for the Correlation Tensors

In order to achieve our goal of obtaining dynamical equations for the integral scales associated with the correlation tensors, we must close these equations in a suitable form. The path that appears most natural is that of requiring that the closed equations contain only the same two-point correlation tensors whose integral scales we seek to determine. This point of view is made plausible by the successes of second-order-closure modeling in that the modeled equations contain only the one-point correlations in question <u>and</u> scales required to give correct dimensionality to the terms that require modeling. If these scales are interpreted as moments of the two-point correlations, no quantity extraneous to the two-point correlations needs to be introduced.

To specify the two-point models more precisely, we adopt two sets of rules. The first set of four is a formalization of the notion of invariant modeling and thus invokes symmetry laws whose validity is hardly in question; the second set summarizes a number of sound dynamical rules whose validity is known to have limitations. The second set of rules is taken merely as a flexible guideline, albeit a very useful one. The first set of rules, the kinematical rules, is as follows:

(1) The model equations are invariant under space rotations, space translations and time translations. These properties are of course true of the unmodeled Eqs. (21), (27), and (33) for R_{ij}, R_j and R. We thus require that the model for a given expression have the same tensor rank and symmetries <u>both</u> in tensor indices and position labels as the expression modeled.

(2) The model equations are invariant under Galilean transformations which is, of course, true of the unmodeled equations.

(3) The modeled equations have the same scaling laws as the unmodeled equations; that is, the model for an expression is to have the same dimensions as the modeled expression.

(4) The modeled expression satisfies the same continuity properties (if any) that are satisfied by the term modeled.

The second set of rules that we adopt intends to capture some of the broad dynamical features of turbulent flows extracted by many workers with much labor over the last fifty years. They are:

(1') The triple correlations represent the diffusive character of turbulent flows, an idea strongly advocated by G. I. Taylor[4] since his early work on turbulence. There are likely exceptions to this rule.

(2') The dissipation occurs at the smaller scale end of the spectrum which is predominantly isotropic. This idea embodies the Kolmogoroff philosophy and, like (1'), this rule has likely exceptions.

(3') The main effect of pressure fluctuations for free flows is to isotropize the turbulence, the Rotta-Batchelor[5,6] concept. We do expect strong exceptions to this rule, for example in the presence of a wall or of strong stratification.

The rules (1'), (2') and (3') have proven very fruitful in second-order closure of one-point correlations. We assume them applicable to the two-point models that we study here.

We have found it very useful to adopt an additional rule, namely that the two-point models should collapse to a standard one-point model when $\underline{r} \to 0$. We shall adhere to this correspondence principle and adopt as standard the second-order modeling of Donaldson and his coworkers (see, for example, Ref. 7).

We consider four main models:

(a) Velocity diffusion. We choose

$$\frac{\partial}{\partial x_k} \langle u_i'(\underline{x})\; u_k'(\underline{x})\; u_j'(\underline{y})\rangle + \frac{\partial}{\partial y_k} \langle u_i'(\underline{x})\; u_k'(\underline{y})\; u_j'(\underline{y})\rangle$$

$$\underset{=}{\mathrm{Mod}} - \frac{\partial}{\partial x_{ck}} \left\{ D_{k\ell}^1 (\underline{x}_c, \underline{r}) \frac{\partial}{\partial x_{c\ell}} R_{ij} + D_{k\ell}^2 (\underline{x}_c, \underline{r}) \left[\frac{\partial R_{\ell j}}{\partial x_{ci}} + \frac{\partial R_{\ell i}}{\partial x_{cj}} \right] \right\} + L_{ij} \tag{58}$$

While this model slightly generalizes previous models by allowing for anisotropic diffusion, we shall be mainly interested in the simple special case

$$D^1_{k\ell} = \frac{1}{3} q(\underline{x}_c)\ \Lambda\ (\underline{x}_c)\delta_{ke} \tag{59}$$

$$D^2_{k\ell} = 0 \tag{60}$$

where Λ is to be a moment of R_{ij} and $q^2 = \langle u_i' u_i' \rangle$. The quantity $L_{ij}(\underline{x}_c, \underline{r})$ is required to have the property

$$L_{ij}\ (\underline{x}_c, \underline{0}) = 0 \tag{61}$$

That is to say, L_{ij} does not contribute to the evolution of the Reynolds stress tensor. Thought of in terms of the theoretical work of the 1940's and 1950's, the term L_{ij} is associated with the cascading of large eddies into small ones. In terms of more recent work where both "merging" of eddies (e.g., in the Brown-Roschko experiments) and cascading can occur, it seems preferable to think of this quantity in terms of eddy-size rearrangement. We shall adopt a simple linear behavior for L_{ij} in terms of R_{ij}. Thus

$$L_{ij} = V \frac{q}{\Lambda}\ [R_{ij} - S_{ij}] \tag{62}$$

where V is a size rearrangement parameter. S_{ij} is required to satisfy

$$S_{ij}\ (\underline{r}=0) = \overline{u_i' u_j'} \tag{63}$$

to guarantee that L_{ij} does not contribute to the evolution of the Reynolds stress and to have vanishing leading moment in order that L_{ij} give a growing contribution to the integral scale. This is required by the experiments on grid turbulence as we shall see in Section 2.

(b) Pressure diffusion. We choose:

$$\frac{\partial}{\partial x_{ci}} \langle p'(\underline{x}) u'_j(\underline{y}) \rangle + \frac{\partial}{\partial x_{cj}} \langle u'_i(\underline{x}) p'(\underline{y}) \rangle \overset{\text{Mod}}{=} - \frac{\partial}{\partial x_{ck}} \quad D^3_{k\ell} \left\{ \frac{\partial R_{\ell j}}{\partial x_{ci}} + \frac{\partial R_{\ell i}}{\partial x_{cj}} \right\} \tag{64}$$

We shall be mainly interested in the special case $D^3 = 0$ which seems adequate at present mainly because pressure and velocity diffusion effects are very difficult to separate

(c) Tendency-towards-isotropy. We choose:

$$\left\langle p'(\underline{x}) \frac{\partial u'_j(\underline{y})}{\partial x_{ci}} + p'(\underline{y}) \frac{\partial u'_i(\underline{x})}{\partial x_{cj}} \right\rangle \overset{\text{Mod}}{=} - \frac{q}{\Lambda} \left[R_{ij} - \delta_{ij} \frac{R_{\alpha\alpha}}{3} + T_{ij} \right] \tag{65}$$

The first two terms in the square bracket are an obvious generalization of the standard Rotta model for the Reynolds stress rate. The tendency correction T_{ij} is required to have vanishing leading moment so that the entire term gives a tendency to isotropy for the scale tensor and furthermore T_{ij} must insure that continuity is fulfilled for the R_{ij} model. It may be shown that for isotropic turbulence T_{ij} is uniquely determined, and it is such as to make the pressure-velocity correlation vector vanish as appropriate to this case. A complete determination of T_{ij} for an arbitrary homogeneous turbulence is not available at present.

(d) Turbulent dissipation. Utilizing our correspondence principle and the guiding rule (2'), we choose (for high Reynolds number)

$$\frac{\partial u'_i(\underline{x})}{\partial x_{ck}} \frac{\partial u'_j(\underline{y})}{\partial x_{ck}} \overset{\text{Mod}}{=} - 2b \frac{q}{\Lambda} \delta_{ij} \frac{R_{\alpha\alpha}}{3} \tag{66}$$

where b = 1/8 (see Ref. 7). This model will be seen in Section 2 to be adequate for near-homogeneous shear turbulence at high (turbulent) Reynolds number.

1.3 Moment Expansion

We propose below a "moment" expansion for functions which are sharply peaked about a definite point which for mathematical discussion can be chosen to be the origin. The terminology is justified on intuitive grounds as well as on the grounds that it can be proven that if a charge density is moment expanded, then the corresponding expansion of the electro-static potential is the conventional multipole expansion. It will be clear that it is necessary to moment expand the two-point tensors if one-point scale equations are to hold. We observe that for any function $f(\underline{x})$ of the vector variable $\underline{x}$, the Dirac identity holds:

$$f(\underline{x}) = \int \delta(\underline{x} - \underline{x}') \, f(\underline{x}') \, d\underline{x}' \tag{67}$$

δ is even in its vector argument and by Schwartz's theory it can be Taylor expanded. According to Schwartz distribution theory, the convergence of the following expressions is understood as convergence in measure. We can thus write the two expansions:

$$\delta(\underline{x} - \underline{x}') = \delta(\underline{x}) - x_i' \frac{\partial}{\partial x_i} \delta(\underline{x}) + \frac{1}{2} x_i' x_j' \frac{\partial^2}{\partial x_i \partial x_j} \delta(\underline{x}) + \ldots \tag{68}$$

and

$$\delta(\underline{x} - \underline{x}') = \delta(\underline{x}' - \underline{x})$$

$$= \delta(\underline{x}') - x_i \frac{\partial}{\partial x_i'} \delta(\underline{x}_i') + \frac{1}{2} x_i x_j \frac{\partial^2}{\partial x_i' \partial x_j'} \delta(\underline{x}') + \ldots \tag{69}$$

Substitution of Eq. (68) into (67) yields

$$f(\underline{x}) = \delta(\underline{x}) \int f(\underline{x}') \, dx' - \frac{\partial \delta(\underline{x})}{\partial x_i} \int x_i' \, f(\underline{x}') \, d\underline{x}' + \ldots \tag{70}$$

which we call the moment expansion of f. Substitution of Eq. (69) into Eq. (67) yields, on the other hand,

$$f(\underline{x}) = f(\underline{0}) - x_i \left(\frac{\partial f}{\partial x_i}\right)_{\underline{x} = \underline{0}} + \cdots \tag{71}$$

which is the standard Taylor expansion of $f(\underline{x})$. The moment expansion, Eq. (70), is suitable for the highly peaked function while the Taylor expansion, Eq. (71), is suited to the opposite regime of slowly varying functions.

A remarkable and useful duality can be proven for the Fourier transformation $\tilde{f}(\underline{k})$:

$$\tilde{f}(\underline{k}) \equiv \int e^{i\underline{k}\cdot\underline{x}} \; f(\underline{x}) \; d\underline{x} \tag{72}$$

Insertion of Eq. (70) into Eq. (72) yields

$$\begin{aligned}\tilde{f}(\underline{k}) &= \int e^{i\underline{k}\cdot\underline{x}} \left[\delta(\underline{x}) \int f(\underline{x}') \; d\underline{x}' - \frac{\partial\delta(\underline{x})}{\partial x_i} \int x_i' \; f(\underline{x}') d\underline{x}' + \cdots \right] d\underline{x} \\ &= \int f(\underline{x}') d\underline{x}' + ik_i \int x_i' \; f(\underline{x}') d\underline{x}' + \cdots \\ &= \tilde{f}(\underline{0}) + k_i \left(\frac{\partial \tilde{f}}{\partial k_i}\right)_{\underline{k}=\underline{0}} + \cdots \end{aligned} \tag{73}$$

where we have used

$$\tilde{f}(\underline{0}) = \int f(\underline{x}') \; d\underline{x}' \tag{74}$$

$$\left(\frac{\partial \tilde{f}}{\partial k_i}\right)_{\underline{k}=\underline{0}} = - \int x_i' \; f(\underline{x}) \; d\underline{x}' \tag{75}$$

which can be deduced from Eq. (72).

Insertion of Eq. (71) into Eq. (72) on the other hand yields

$$\tilde{f}(\underline{k}) = \int e^{i\underline{u}\cdot\underline{x}} \left[f(\underline{0}) - x_i \left(\frac{\partial f}{\partial k_i}\right)_{\underline{x}=0} + \cdots \right] d\underline{x} =$$

$$= (2\pi)^3\ \delta(\underline{k})\ f(\underline{0}) + i(2\pi)^3\ \frac{\partial\delta(\underline{k})}{\partial k_i}\ \left(\frac{\partial f}{\partial x_i}\right)_{x=0} + \ldots$$

$$= \delta(\underline{k}) \int \tilde{f}(\underline{k}')\ dk' - \frac{\partial\delta(\underline{k})}{\partial k_i} \int k_i'\ \tilde{f}\ (\underline{k}')\ dk + \ldots \tag{76}$$

where we have used

$$f(\underline{0}) = \int \tilde{f}(\underline{u}')\ d\underline{k}'/(2\pi)^3 \tag{77}$$

$$\left(\frac{\partial f}{\partial x_i}\right)_{\underline{x}=0} = - \int k_i'\ \tilde{f}(\underline{k}')\ d\underline{k}'/(2\pi)^3 \tag{78}$$

which follow easily from the inverse of Eq. (72).

We may now expand the tensors R_{ij}, R_i, and R in terms of their moments. From dimensional considerations the moments necessarily define lengths. A simple choice was suggested by Rotta (Ref. 2) for the velocity correlation tensor. This choice was alluded to earlier in this paper and is

$$M_{ij}^{(2)} = \int d\underline{r}\ \frac{R_{ij}(\underline{x}_c, \underline{r})}{4\pi r^2} = M_{ij}^{(2)}(x_c) \tag{79}$$

The identification (suggested by Rotta) of the lengths contained in $M_{ij}^{(2)}$ is

$$M_{ij}^{(2)} = \frac{q^2(\underline{x}_c)}{3}\ \Lambda_{ij}(\underline{x}_c) \tag{80}$$

where, of course

$$q^2(\underline{x}_c) = \lim_{\underline{r}\to\underline{0}}\ R_{kk}(\underline{x}_c, \underline{r}) \tag{81}$$

An alternative identification, convenient for certain purposes (i.e., the homogeneous flows of Section 2), is

$$M_{ij}^{(2)} = \overline{u_i'u_j'}\, L \tag{82}$$

which has the same dimensionality but a somewhat different tensor character than the Rotta version, Eq. (80). We adopt in analogy with Eq. (78)

$$M_i^{(1)} (\underline{x}_c) = \int d\underline{r} \frac{R_i(\underline{x}_c, \underline{r})}{4\pi r^2} \tag{83}$$

and

$$M^{(0)} (\underline{x}_c) = \int d\underline{r} \frac{R(\underline{x}c, \underline{r})}{4\pi r^2} \tag{84}$$

The analogues of the identification, Eq. (80), for the velocity scale may be chosen to be

$$M_i^{(1)} (\underline{x}_c) = \frac{1}{3} q(\underline{x}_c) (\overline{\theta'^2})^{1/2} \Lambda_i^{(1)} (\underline{x}_c) \tag{85}$$

and

$$M^{(0)} (\underline{x}_c) = \overline{\theta'^2}\, \Lambda^{(0)} (\underline{x}_c) \tag{86}$$

The leading term in the moment expansion, Eq. (70), can then be taken to be

$$R_{ij}(\underline{x}_c, \underline{r}) = M_{ij}^{(2)} (\underline{x}_c)\, \delta(\underline{r}) \tag{87}$$

where we have introduced the one-dimensional Dirac function $\delta(r)$ (r is defined as the magnitude of the vector $\underline{r}$, $r = \underline{r}$). The main property of $\delta(r)$, for the present purposes, is its relation to $\delta(\underline{r})$. This is given by

$$\delta(\underline{r}) = \frac{\delta(r)}{4\pi r^2} \tag{88}$$

We also introduce the moment expansion for the correlation R_i and R as follows

$$R_i(\underline{x}_c, \underline{r}) = M_i^{(1)} (\underline{x}_c)\ \delta(r) \tag{89}$$

and

$$R(\underline{x}_c, \underline{r}) = M^{(0)} (\underline{x}_c)\ \delta(r) \tag{90}$$

In order to obtain the rate equations for the moments, we require three steps which we are now in a position to take:

(a) Develop a closure model for the R_{ij} equation so that when $\underline{y} \to \underline{x}$, we obtain a standard equation for $\overline{u_i'u_j'}$. This is a useful correspondence criterion designed to allow us to incorporate what has already been gained from second-order closure. Of course, what is said for R_{ij} is taken to apply to R_i and to R as well.

(b) Substitute into the closed equations for the two-point correlation tensors, R_{ij}, R_i and R, their moment expansions and retain the leading term to define a first approximation to the behavior of these quantities. We observe that the moment expansion appears here as an intermediate step, prior to integration over the variable $\underline{r}$ with respect to which we wish to average. We return to this essential point below (c).

(c) We integrate over all $\underline{r}$ the first-order expanded model equations for the correlation tensors. This step of integration (or averaging) is taken in many known moment procedures <u>without</u> the intermediate step (b). We return below to this point to clarify our position.

At this juncture we believe that two main observations are worth making. With respect to (a), we observe that in order to have a closed set of partial differential equations which are supplemented <u>only</u> by boundary and initial conditions we must obtain closure at the level of the two-point correlation tensors. This is true provided we identify the spatial scales introduced in the (one-point) second-order closure equations as integral scales associated with two-point correlation functions when we utilize the correspondence principle expressed in (a). We are cognizant that such a procedure is a strong assumption. Hence, our label: "generalized second-order closure."

The second point that we wish to make refers to the requirement of an intermediate step, the moment expansion in (b), prior to the development of the moment equations. This step is the least orthodox in our development, and we have repeatedly attempted to bypass it. We believe that the moment expansion is necessary for the development of rate equations for the scales. Reconsider briefly, as a prototype, the standard procedure, due to Maxwell, utilized in obtaining moment equations from the Boltzmann equations (for the single-particle velocity distribution function in a gas) on the way to deriving the transport coefficients for the Navier-Stokes equations. We consider here velocity space moments of the quantity

$$\delta f \equiv \frac{\partial}{\partial t} f(\underline{v},t) + \underline{v}\cdot\underline{\nabla} f - J[f,f] \tag{91}$$

where J is the collision integral, bilinear in f. That is, we multiply Eq. (91) by powers of $\underline{v}$ and then integrate over all velocity space. We observe that: (i) the convection term is linear in the averaged variable $\underline{v}$ and hence raises the moment index by precisely one, and (ii) the first five moments of J vanish as a consequence of conservation of mass, momentum and kinetic energy. An orderly hierarchy of moments ensues, with the moment equations for momentum and for energy containing exactly one higher moment each. The closure of these equations and hence the calculation of these two higher moments is then the task of the Chapman-Enskog expansion. Feature (i) is characteristic of all the moment procedures known to the writers, and it is essential for the implementation of Maxwell's approach. The absence of this feature in the equations for the correlation tensors of turbulent fluids is what forces our intermediate step, the moment expansion. For the correlation tensors, convection occurs independently at two distinct points. By contrast, the integral scales evolve at one (and each) point of space since they are functions of the single variable $\underline{x}_c$. We submit that the reduction of the two points $\underline{x}$ and $\underline{y}$ to the single $\underline{x}_c$ cannot be accomplished without our moment expansion, Eq. (70).

The reader can rather easily convince himself, in the cases where the intermediate step (b) of carrying out the moment expansion is not necessary (or in the case described above of Maxwell moments of the Boltzmann equation), that the explicit use of step (b) causes no deviations at all from the standard procedures.

When we carry out the program contained in (a), (b) and (c) above, we obtain the sought-after rate equations for the moments (at the single point $\underline{x}_c$). We illustrate the procedure with the simplest term, the partial time derivative of R_{ij}. Application of the procedure to other terms in the two-point correlation equation is tedious but not difficult.

(a) Consider the relevant term in the modeled equation. The term is

$$T_{ij}^{(3)} \equiv \frac{\partial}{\partial t} R_{ij} (\underline{x}_c, \underline{r}) \tag{92}$$

For this term, of course, no modeling was necessary.

(b) Insert in the term (modeled or not) the moment expansion and retain the leading term

$$T_{ij}^{(3)} = \frac{\partial}{\partial t} \left[M_{ij}^{(2)} (\underline{x}_c, t) \ \delta(r) \right] = \delta(r) \ \frac{\partial M_{ij}^{(2)}}{\partial t} \tag{93}$$

where we have used Eq. (87). It may not be amiss to remark that a full series expansion can be considered even though for simplicity we focus here on the leading term.

(c) Integrate the quantity obtained in (b) over all space of relative position $\underline{r}$ using a weight appropriate to the generalized moment considered. Using the Rotta moment defined in Eq. (79), we consider

$$\int \frac{d\underline{r}}{4\pi r^2} T_{ij}^{(3)} = \int \frac{d\underline{r}}{4\pi r^2} \left[\delta(r) \ \frac{\partial}{\partial t} M_{ij}^{(2)} (\underline{x}_c, t) \right] =$$

$$= \frac{\partial}{\partial t} M_{ij}^{(2)}(\underline{x}_c,t) \int \frac{d\underline{r}}{4\pi r^2}\,\delta(r)$$

$$= \frac{\partial}{\partial t} M_{ij}^{(2)}(\underline{x}_c,t) \int d\underline{r}\;\delta(\underline{r})$$

$$= \frac{\partial}{\partial t} M_{ij}^{(2)}(\underline{x}_c,t) \tag{94}$$

: used the basic property, Eq. (88), of $\delta(r)$ and the familiar

$$\int d\underline{r}\;\delta(\underline{r}) = 1 \tag{95}$$

see that implementing steps (a), (b) and (c) yields rate
· the moments.

general formula can be given for averaging with a function A
ariable which can be the component of a tensor

$$\int \frac{d\underline{r}}{4\pi r^2}\; A(\underline{x})\left[\frac{\partial R_{ij}}{\partial x_\ell} + A(\underline{y})\;\frac{\partial R_{ij}}{\partial y_\ell}\right]$$

$$= A(\underline{x}_c)\;\frac{\partial}{\partial x_{c\ell}} M_{ij}^{(2)}\;(\underline{x}_c) - \frac{1}{3} M_{ij}^{(2)}\;(\underline{x}_c)\;\frac{\partial A}{\partial x_{c\ell}} \tag{96}$$

der as a complete example the following model equations in
mplicity, diffusion terms have been omitted. We thus consider
neous flows at high turbulent Reynolds number. For the
ocity correlation tensor we assume

$$\left(R_{ik}\;\frac{\partial \bar{u}_j}{\partial y_k} + \frac{\partial \bar{u}_i}{\partial x_k}\;R_{kj}\right) + \nu\;\frac{q}{\Lambda}\,[R_{ij} - S_{ij}] +$$

$$- \frac{q}{\Lambda}\left[R_{ij} - \delta_{ij}\frac{R_{\alpha\alpha}}{3} + T_{ij}\right] - 2b\,\frac{q}{\Lambda}\,\delta_{ij}\,\frac{R_{\alpha\alpha}}{3} \tag{97}$$

For the two-point flux of the passive additive we assume

$$\frac{DR_i}{Dt} = - R_{ik}\frac{\partial\bar{\theta}}{\partial y_k} - \frac{\partial\bar{u}_i}{\partial x_k} R_k + v'\,\frac{q}{\Lambda}[R_i - S_i] - A\,\frac{q}{\Lambda}[R_i - T_i] \tag{98}$$

and for the two-point autocorrelation function of the passive additive (e.g., temperature autocorrelation), we assume

$$\frac{DR}{Dt} = - \left[R_k \frac{\partial\bar{\theta}}{\partial y_k} + \frac{\partial\bar{\theta}}{\partial x_k} R_k\right] - 2bs\,\frac{q}{\Lambda}\,R \tag{99}$$

where the two-point material derivative D/Dt is defined by

$$\left(\frac{D}{Dt}\right)_{\substack{\text{two-}\\ \text{point}}} \equiv \frac{\partial}{\partial t} + \bar{u}_k(\underline{x})\,\frac{\partial}{\partial x_k} + \bar{u}_k(\underline{y})\,\frac{\partial}{\partial y_k} \tag{100}$$

The model parameters v, v', A, b and s will be discussed in Section 2. In addition to Eqs. (97), (98) and (99), we consider the requirements of continuity:

$$\frac{\partial R_{ij}}{\partial x_i} = 0 \tag{101}$$

$$\frac{\partial R_{ij}}{\partial y_j} = 0 \tag{102}$$

$$\frac{\partial R_i}{\partial x_i} = 0 \tag{103}$$

We then obtain, from the two-point models given, the following one-point correlation equations by taking the limit $\underline{r} \to \underline{0}$. The kinematic Reynolds stress $\langle u_i' u_j' \rangle$ satisfies

$$\frac{D\overline{u_i' u_j'}}{Dt} = - \left[\overline{u_i' u_k'} \frac{\partial \bar{u}_j}{\partial x_{ck}} + \overline{u_j' u_k'} \frac{\partial \bar{u}_i}{\partial x_{ck}} \right] - \frac{q}{\Lambda} \left[\overline{u_i' u_j'} - \delta_{ij} \frac{q^2}{3} \right] - \delta_{ij} \frac{2bq^3}{3\Lambda} \tag{104}$$

The flux of passive additive satisfies

$$\frac{D\overline{u_i' \theta'}}{Dt} = - \overline{u_i' u_k'} \frac{\partial \bar{\theta}}{\partial x_{ck}} - \overline{u_k' \theta'} \frac{\partial \bar{u}_i}{\partial x_{ck}} - A \frac{q}{\Lambda} \overline{u_i' \theta'} \tag{105}$$

and finally the temperature variance satisfies

$$\frac{D}{Dt} \overline{\theta'^2} = - 2 \overline{u_k' \theta'} \frac{\partial \bar{\theta}}{\partial x_{ck}} - 2bs \frac{q}{\Lambda} \overline{\theta'^2} \tag{106}$$

We have used, in the one-point correlation equations, Eqs. (104), (105) and (106), the one-point material derivative

$$\left(\frac{D}{Dt} \right)_{\substack{\text{one-} \\ \text{point}}} \equiv \frac{\partial}{\partial t} + \bar{u}_k (\underline{x}_c) \frac{\partial}{\partial x_{ck}} \tag{107}$$

omitting the distinction between Eqs. (100) and (107) since no confusion can arise. The equations (104) through (106) are those of our standard model as given by Lewellen[7].

When we apply to the two-point equations, (97), (98), and (99), the moment expansion and average procedure specified above, we obtain the moment equations. A tensor scale is contained in the velocity moment equation

$$\begin{aligned} \frac{D}{Dt} M_{ij}^{(2)} &= - M_{ik}^{(2)} \frac{\partial \bar{u}_j}{\partial x_{ck}} - \frac{\partial \bar{u}_i}{\partial x_{ck}} M_{kj}^{(2)} + v \frac{q}{\Lambda} M_{ij}^{(2)} \\ &\quad - \frac{q}{\Lambda} \left[M_{ij}^{(2)} - \frac{\delta_{ij}}{3} M_{kk}^{(2)} \right] - 2bq^3 \delta_{ij} \end{aligned} \tag{108}$$

A vector scale is determined by the moment of the flux which satisfies

$$\frac{D}{Dt} M_i^{(1)} = - M_{ik}^{(2)} \frac{\partial \bar{\theta}}{\partial x_{ck}} - M_k^{(1)} \frac{\partial \bar{u}_i}{\partial x_{ck}} - (A + \nu') \frac{q}{\Lambda} M_i^{(1)} \tag{109}$$

And finally the temperature scale is determined by

$$\frac{D}{Dt} M^{(0)} = - 2 M_k^{(1)} \frac{\partial \bar{\theta}}{\partial x_{ck}} - 2bs \frac{q}{\Lambda} M^{(0)} \tag{110}$$

We shall see in Section 2 that, accepting the identification of the scales given in Eq. (80), we obtain good agreement with experiments on near-homogeneous turbulent shear flows.

2. NEARLY HOMOGENEOUS FLOWS

The goal of this section is to demonstrate that the theory of turbulent scales proposed in the previous section is in considerable agreement with the experiments carried out at JHU/APL on nearly homogeneous flows.

To introduce our discussion of nearly homogeneous flows, we observe some general features of the scale equations that we have derived and then will exhibit exact solutions for theoretically homogeneous flows. Adopting our definition of the tensor integral scale for velocity components that we discussed in the previous section, namely,

$$\frac{q^2}{3}\Lambda_{ij}(\underline{x}_c) = \int \frac{d\underline{r}}{4\pi r^2} R_{ij}(\underline{x}_c,\underline{r}) \tag{111}$$

the moment equation for $M_{ij}^{(2)}$ can then be written as (we drop the subscript c)

$$\begin{aligned}\frac{\partial}{\partial t}\left(q^2\Lambda_{ij}\right) + \bar{u}_k \frac{\partial}{\partial x_k}\left(q^2\Lambda_{ij}\right) &= -q^2\left(\Lambda_{ij}\frac{\partial \bar{u}_j}{\partial x_k} + \frac{\partial \bar{u}_i}{\partial x_k}\Lambda_{kj}\right)\\ &+ \nu\frac{\partial}{\partial x_k}\left[(q\Lambda)\frac{\partial}{\partial x_k}\left(q^2\Lambda_{ij}\right)\right] + \nu'\frac{q}{\Lambda}\left(q^2\Lambda_{ij}\right)\\ &- \frac{q^3}{\Lambda}\left(\Lambda_{ij} - \frac{\delta_{ij}}{3}\Lambda_{kk}\right) - 2bq^3\delta_{ij}\end{aligned} \tag{112}$$

As indicated in the Introduction, it is promising to identify the scale Λ that enters the modeled terms with

$$\Lambda \equiv \frac{1}{3}\Lambda_{kk} \tag{113}$$

We can contract Eq. (112) and subtract the rate of q^2 to obtain a local rate equation for Λ . This reads

$$\frac{\partial}{\partial t}\Lambda + \bar{u}_k \frac{\partial \Lambda}{\partial x_k} = -2\Lambda\left(\frac{\Lambda_{ij}}{3\Lambda} - \frac{\langle u_i'u_j'\rangle}{q^2}\right)\frac{\partial \bar{u}_i}{\partial x_j}$$

$$+\left(\frac{\nu}{q^2}\right)\frac{\partial}{\partial x_k}\left[(q\Lambda)\frac{\partial}{\partial x_k}(q^2\Lambda)\right] - \left(\frac{\nu\Lambda}{q^2}\right)\frac{\partial}{\partial x_k}\left[(q\Lambda)\frac{\partial q^2}{\partial x_k}\right]$$

$$+ \nu' q \tag{114}$$

It is of interest that the coefficient of $\partial\bar{u}_i/\partial x_j$ in the first term on the right-hand side of Eq. (114) is not proportional to $\overline{u_i'u_j'}$. Such as assumption has been made repeatedly in the universal "scalar scale" models. A simple approximation, that we call the σ approximation, can be deduced from the assumption

$$\Lambda_{ij} = \Lambda\left[\delta_{ij} + \sigma\left(\frac{\langle u_i'u_j'\rangle}{q^2} - \frac{\delta_{ij}}{3}\right)\right] \tag{115}$$

Substituting the ansatz Eq. (115) into Eq. (114), the "production term" simplifies, and we find

$$\frac{\partial \Lambda}{\partial t} + \bar{u}_k \frac{\partial \Lambda}{\partial x_k} = -2\left(\frac{\sigma}{3} - 1\right)\frac{\Lambda\langle u_i'u_j'\rangle}{q^2}\frac{\partial \bar{u}_i}{\partial x_j}$$

$$+\left(\frac{\nu}{q^2}\right)\frac{\partial}{\partial x_k}\left[(q\Lambda)\frac{\partial}{\partial x_k}(q^2\Lambda)\right] - \left(\frac{\nu\Lambda}{q^2}\right)\frac{\partial}{\partial x_k}\left[(q\Lambda)\frac{\partial q^2}{\partial x_k}\right]$$

$$+ \nu' q \tag{116}$$

Furthermore, we can obtain an equation for σ by substituting the ansatz Eq. (115) into (112), subtracting the rate of change of Λ given by Eq. (116) and contracting the resulting second-rank tensor equation by the deviator $\langle u_i'u_j'\rangle - (1/3)\delta_{ij}q^2$ and solving for $\partial\sigma/\partial t$. The resulting σ equation is

$$\frac{\partial\sigma}{\partial t} + \bar{u}_k \frac{\partial\sigma}{\partial x_k} = 2\left(\frac{\sigma}{3} - 1\right)\left(\sigma + \frac{q^4}{\langle u_i'u_j'\rangle\langle u_i'u_j'\rangle - \frac{q^4}{3}}\right) \frac{\langle u_k'u_\ell'\rangle}{q^2} \frac{\partial\bar{u}_k}{\partial x_\ell}$$

$$+\left(\frac{\nu}{\Lambda}\right) \frac{\partial}{\partial x_k} \left[(q\Lambda) \frac{\partial}{\partial x_k} (\sigma\Lambda)\right] + \left(\frac{\nu\sigma}{\Lambda}\right) \frac{\partial}{\partial x_k} \left[(q\Lambda) \frac{\partial\Lambda}{\partial x_k}\right] \qquad (117)$$

The absence of nondiffusive equilibrium for Λ as given in Eq. (116) is by no means accidental. It is, in fact, essential for a good match of the model parameters to the nearly homogeneous flow data (both for simple grid and for near-homogeneous shears). We note from the σ equation, as pointed out by Donaldson and Sandri[8], that flows with different turbulent structure have a different coefficient $(\sigma/3) - 1$ in front of the Λ scale "production" term. Thus a scalar scale equation which does not take into account the turbulent structure cannot be valid. Furthermore, we can prove that

$$F \equiv \langle u_i'u_j'\rangle\langle u_i'u_j'\rangle - \frac{q^4}{3} \geq 0 \qquad (118)$$

quite generally, with

$$q^2 = \langle u_k'u_k'\rangle \qquad (119)$$

Therefore, the only non-negative stationary solution of Eq. (117) which is nondiffusive is $\sigma = 3$. We shall see below that all the solutions for homogeneous shear asymptote to a solution that can be exhibited exactly and for which $\sigma = 3$. It is interesting that the value of σ used in practice (in the universal scalar scale modeling) is $\sigma = 2.48$. As pointed out again by Donaldson and Sandri[8], it is attractive to conjecture that diffusion always lowers the value of σ from its asymptotic value for homogeneous flows and that the different behavior of two-dimensional and axisymmetric jets can be attributed to the substantial differences in the diffusion terms in the two geometries.

We now present two exact solutions for ideally homogeneous shears. We start with some notation.

The centroid vector and mean velocity vector are taken to have components

$$(x,y,z) \quad , \quad (U(y),0,0)$$

with $\partial U/\partial y = U' =$ constant. The relevant components of the Reynolds stress equations are obtained from Eq. (46). We drop primes on the fluctuations and give a form useful for numerical integration in which $\overline{u_1^2}$ and Λ_{11} are calculated from

$$\overline{u_1^2} = \overline{q^2} - \overline{u_2^2} - \overline{u_3^2} \tag{120}$$

$$\Lambda_{11} = 3\Lambda - \Lambda_{22} - \Lambda_{33} \tag{121}$$

The other relevant components of the stress and energy equations are

$$\frac{\partial}{\partial t} \overline{u_2^2} = \frac{1}{3} (1 - 2b) \frac{q^3}{\Lambda} - \frac{q}{\Lambda} \overline{u_2^2} \tag{122}$$

$$\frac{\partial}{\partial t} \overline{u_3^2} = \frac{1}{3} (1 - 2b) \frac{q^2}{\Lambda} - \frac{q}{\Lambda} \overline{u_3^2} \tag{123}$$

$$\frac{\partial}{\partial t} \overline{u_1 u_2} = -\overline{u_2^2}\, U' - \frac{q}{\Lambda} \overline{u_1 u_2} \tag{124}$$

$$\frac{\partial q}{\partial t} = - \frac{\overline{u_1 u_2}}{q} U' - \frac{bq^2}{\Lambda} \tag{125}$$

For the tensor scale components, we obtain, using Eq. (112),

$$\frac{\partial}{\partial t} \Lambda_{22} = - \frac{1}{T} \Lambda_{22} + (1 - 2b)q \tag{126}$$

$$\frac{\partial}{\partial t} \Lambda_{33} = - \frac{1}{T} \Lambda_{33} + (1 - 2b)q \tag{127}$$

$$\frac{\partial}{\partial t}\Lambda_{12} = -\frac{1}{T}\Lambda_{12} - U'\Lambda_{22} \tag{128}$$

$$\frac{\partial}{\partial t}\Lambda = 2\,\frac{\overline{u_1u_2}}{q^2}\,U'\Lambda - \frac{2}{3}\Lambda_{12}U' + v'q \tag{129}$$

where

$$\frac{1}{T} = -2\,\frac{\overline{u_1u_2}}{q^2}\,U' + \frac{q}{\Lambda}(1 - 2b - v') \tag{130}$$

2.1 Solution of the Shearless Equations

Setting $U' = 0$, we see that equations for q and Λ decouple from the tensor components. Introducing the deviators

$$d_{ij} = \overline{u_iu_j} - \frac{1}{3}\delta_{ij}q^2 \tag{131}$$

$$D_{ij} = \Lambda_{ij} - \frac{1}{3}\delta_{ij}\Lambda_{kk} \tag{132}$$

and the time

$$\tau = \frac{\Lambda}{q} \tag{133}$$

we have the set

$$\frac{\partial}{\partial t}q = -\frac{b}{\tau}q \quad , \quad \frac{\partial}{\partial t}\Lambda = \frac{v'}{\tau}\Lambda \tag{134}$$

$$\frac{\partial}{\partial t}d_{ij} = -\frac{b}{\tau}d_{ij} \, , \, \frac{\partial}{\partial t}D_{ij} = -\frac{1-2b-v'}{\tau}D_{ij} \tag{135}$$

$$\frac{\partial}{\partial t}\tau = b + v' \tag{136}$$

Integrating Eq. (136) as

$$\tau = (b+v')(t-t_0) + \frac{\Lambda_0}{q_0} \tag{137}$$

we see that q , Λ , d_{ij} and D_{ij} are suitable powers of $(q_0/T_0)\tau$; for example,

$$q = q_o \left[(b + v') \frac{q_o}{\Lambda_o} (t - t_o) + 1 \right]^{-b/(b+v')} \tag{138}$$

$$d_{ij} = d_{ij_o} \left[(b + v') \frac{q_o}{\Lambda_o} (t - t_o) + 1 \right]^{-1/(b+v')} \tag{139}$$

$$\Lambda = \Lambda_o \left[(b + v') \frac{q_o}{\Lambda_o} (t - t_o) + 1 \right]^{-v'/(b+v')} \tag{140}$$

A good fit to experimental data on the decay of turbulent energy and the growth of eddy size for grid turbulence is obtained if one chooses

$$\begin{aligned} b &\approx \frac{1}{8} \\ v' &= 0.075 \end{aligned} \tag{141}$$

We then see that for large times

$$q^2 \sim q_o^2 \left[\cdot 2 \frac{q_o}{\Lambda_o} t \right]^{-5/4} \tag{142}$$

$$d_{ij} \sim d_{ij_o} \left[\cdot 2 \frac{q_o}{\Lambda_o} t \right]^{-5} \tag{143}$$

which shows that the deviator decays with a power about four times larger than the energy.

From the solutions given above, we can verify that statistics are preserved by the model equations if the model parameters satisfy certain bounds. We first show that the two tensors $\overline{u_i u_j}$ and Λ_{ij} are positive definite from their definitions. Consider an arbitrary (constant) A_i then,

$$A_i \overline{u_i u_j} A_j = \overline{(u \cdot A)^2} \geq 0 \tag{144}$$

the equality sign holding for $A \equiv 0$ only. Thus, $\overline{u_i u_j}$ is a positive definite tensor.

From the definition of the scale tensor given by Eq. (111), using Fourier transform on R_{ij} ,

$$\frac{q^2}{3}\Lambda_{ij} = \int \frac{R_{ij}}{4\pi r^2}\, d\vec{r} = \int \frac{\phi_{ij}}{8\pi k}\, d\vec{k} \tag{145}$$

where the power spectrum tensor ϕ_{ij} is positive definite by Khiutchine's theorem. Thus,

$$\frac{q^2}{3} A_i \Lambda_{ij} A_j = \quad \frac{d\vec{k}}{8\pi k} A_i \phi_{ij} A_j \geq 0 \tag{146}$$

Thus, Λ_{ij} is positive definite because q^2 is positive as a consequence of Eq. (144).

Using the solution (141) and an analogous solution for Λ_{ij} , we find

$$\begin{aligned}\overline{u_i u_j}(t) = \overline{u_i u_j}(0)\left(\frac{q_o}{\Lambda_o}\tau\right)^{-1/(b+v')} \\ + \frac{1}{3}\delta_{ij} q_o^2\left[\left(\frac{q_o}{\Lambda_o}\tau\right)^{-2b/(b+v')} - \left(\frac{q_o}{\Lambda_o}\tau\right)^{-1/(b+v')}\right]\end{aligned} \tag{147}$$

$$\begin{aligned}\Lambda_{ij}(t) = \Lambda_{ij}(0)\left(\frac{q_o}{\Lambda_o}\tau\right)^{-(1-2b-v')/(b+v')} \\ + \Lambda(0)\delta_{ij}\left[\left(\frac{q_o}{\Lambda_o}\tau\right)^{v'/(b+v')} - \left(\frac{q_o}{\Lambda_o}\tau\right)^{-(1-2b-v')/(b+v')}\right]\end{aligned} \tag{148}$$

We now multiply Eq. (147) by $A_i A_j$ when A_i is an arbitrary vector, and find

$$A_i\overline{u_iu_j}(t)A_j = \overline{(A \cdot u)^2}(0)\left(\frac{q_o}{\Lambda_o}\tau\right)^{-1/(b+v')}$$

$$+ \frac{A^2}{3} q_o^{\,2}\left[\left(\frac{q_o}{\Lambda_o}\tau\right)^{-2b/(b+v')} - \left(\frac{q_o}{\Lambda_o}\tau\right)^{-1/(b+v')}\right] \tag{149}$$

From (137) we see that

$$\frac{q_o}{\Lambda_o}\tau \geq 1 \qquad (b + v' \geq 0) \tag{150}$$

Sufficient for the left-hand side of Eq. (149) to be positive is

$$\left(\frac{q_o}{\Lambda_o}\tau\right)^{-2b/(b+v')} \geq \left(\frac{q_o}{\Lambda_o}\tau\right)^{-1/(b+v')} \tag{151}$$

which requires, using Eq. (150)

$$2b \leq 1 \tag{152}$$

A similar analysis applies to Λ_{ij} ; however, no further restrictions on the parameters are found.

2.2 An Asymptotic Solution for the Equations with Shear

To obtain a solution of the equations with shear, we let

$$\begin{aligned}
q &= V\, e^{aU't} & \Lambda &= L\, e^{aU't} \\
\overline{u_1^2} &= W_1\, e^{2aU't} & \Lambda_{11} &= L_1\, e^{aU't} \\
\overline{u_2^2} &= W_2\, e^{2aU't} & \Lambda_{22} &= L_2\, e^{aU't} \\
\overline{u_3^2} &= W_3\, e^{2aU't} & \Lambda_{33} &= L_3\, e^{aU't} \\
\overline{u_1u_2} &= W_4\, e^{2aU't} & \Lambda_{12} &= L_4\, e^{aU't}
\end{aligned} \tag{153}$$

Substituting these forms into the differential equations, we find that the exponentials cancel and that an algebraic set of equations for the amplitudes is obtained. It is possible, with some algebra, to solve the amplitude equations explicitly in terms of the parameters b and v'. The energy components are

$$\frac{\overline{u_1^2}}{q^2} = \frac{1 + 6v' + 4b}{3(1 + 2v')} = 0.5652 \tag{154}$$

$$\frac{\overline{u_2^2}}{q^2} = \frac{\overline{u_3^2}}{q^2} = \frac{1 - 2b}{3} \frac{1}{1 + 2v'} = 0.2174 \tag{155}$$

The scale components are

$$\frac{\Lambda_{11}}{\Lambda} = \frac{1 + 6v' + 4b}{1 + 2v'} = 1.696 \tag{156}$$

$$\frac{\Lambda_{22}}{\Lambda} = \frac{\Lambda_{33}}{\Lambda} = \frac{1 - 2b}{1 + 2v'} = 0.652 \tag{157}$$

We see that

$$\frac{\Lambda_{11}}{\Lambda_{22}} = \frac{\overline{u_1^2}}{\overline{u_2^2}} = \frac{1 + 6v' + 4b}{1 - 2b} = 2.60 \tag{158}$$

The off-diagonal components are

$$Br = \frac{|\overline{u_1 u_2}|}{q^2} = \frac{1}{1 + 2v'} \sqrt{\frac{(1 - 2b)(b + v')}{3}} = 0.194 \tag{159}$$

$$\frac{\Lambda_{12}}{\Lambda} = - \frac{1}{1 + 2v'} \sqrt{3(1 - 2b)(b + v')} = -0.194 \tag{160}$$

The Corrsin parameter is

$$Co \equiv \frac{\overline{u_1 u_2}}{\sqrt{\overline{u_1^2}\ \overline{u_2^2}}} = \sqrt{\frac{3(b + v')}{1 + 4b + 6v'}} = 0.55 \tag{161}$$

The ratio of the two times is

$$\frac{1}{\alpha} = \frac{q}{U'\Lambda} = \frac{1}{1 + 2v'} \sqrt{\frac{1 - 2b}{3(b + v')}} = 0.972 \tag{162}$$

and the growth rate, a , is

$$a = v' \frac{1}{\alpha} = \frac{1}{1 + 2v'} \sqrt{\frac{1 - 2b}{3(b + v')}} = 0.073 \tag{163}$$

We notice two additional interesting parameters:

$$\frac{|\overline{u_1 u_2}|}{q^2} \frac{U'}{q} = \alpha \cdot Br = b + v' = 0.20 \tag{164}$$

$$\frac{\Lambda_{ij}}{\Lambda} - \delta_{ij} = 3\left(\frac{\overline{u_i u_j}}{q^2} - \frac{1}{3}\delta_{ij}\right) \tag{165}$$

In terms of the σ parameter, Eq. (165) corresponds exactly to the value

$$\sigma = 3 \tag{166}$$

As remarked above, $\sigma = 2.48$ has given satisfactory results on a number of flows. Also note that for this coupled, asymptotic, convective solution, the ratios $\langle u_i u_j \rangle / q^2$, $q/\Lambda U'$ are modified from those of the equilibrium solution with no diffusion (the superequilibrium introduced by Donaldson).[9] For these solutions, we have (using Eq. (46)),

$$\frac{\langle uu \rangle}{q^2} = \frac{1 + 4b}{3} = 0.50 \tag{167}$$

$$\frac{\langle vv \rangle}{q^2} = \frac{\langle ww \rangle}{q^2} = \frac{1 - 2b}{3} = 0.25 \tag{168}$$

$$\frac{\langle uv \rangle}{q^2} = b\sqrt{\frac{1 - 2b}{3b}} = 0.177 \tag{169}$$

$$\frac{q}{\Lambda(\partial U/\partial y)} = \frac{1 - 2b}{3b} = 2 \tag{170}$$

now appropriate that we consider the agreement between these results just obtained and experimental measurements. The meas- will make use of are those of Harris, Graham and Corrsin (Ref.10). re is Figure 3 from Reference 10. It shows the growth in $\langle u_1^2 \rangle$, $\langle u_3^2 \rangle$ in a constant shear flow. For these measurements, U_c /sec, $U' = \partial U/\partial y = 48 \text{ sec}^{-1}$ and $h = 30.48$ cm. Also shown in is the behavior of the Corrsin number

$$- \langle u_1 u_2 \rangle \Big/ \sqrt{\langle u_1^2 \rangle \langle u_2^2 \rangle}$$

pect, after an initial transient, that the flow adjusts to an solution with certain parameters constant, we plot faired results 3 in semi-logarithmic form in Figure 4. It is apparent from at an exponential solution has been reached at an x/h of approx- We note that the growth rates in terms of x/h may be written

$$\langle u^2 \rangle \quad , \quad \langle u_2^2 \rangle \, , \langle u_3^2 \rangle \propto e^{2a_c x/h} \tag{171}$$

re we have indicated the a_c's that best fit each curve. An these values gives $a_c = 0.087$ or, let us say, $a_c = 0.09$. To se results with the theoretical results just given, we must multi- $U_c/hU' = 0.848$ which gives $a = 0.076$. This is not in bad ith the theoretical value of 0.0729.

am has been written to solve the full set of coupled $\langle u_i u_j \rangle$ and ons, namely, Eqs. (46) and (112), and the result of a computa- versus both dimensional time and nondimensional time

$$U'\Delta t = \left(\frac{x}{h} - \left(\frac{x}{h}\right)_0\right)\frac{hU'}{U_c} \tag{172}$$

Figure 5. It is seen that, although the growth rate of q is ed, the actual values of q are some 20% in error and remain so

in the asymptotic region of the solution. This is not good, but we will discuss a possible cause of this error after we have exhibited the agreement between experimental results and all the parameters we have derived for the exponential behavior of a homogeneous shear flow. These comparisons are shown in the second, third and fourth columns of Table 1.

It is not difficult to show that other simpler, second-order models of turbulent flow have exponential asymptotic solutions in the case of homogeneous shear flow. In Table 1, we also show the theoretical results for the following models:

(a) A full closure in the case of the $\langle u_i u_j \rangle$ tensor and the single scale equation

$$\frac{d\Lambda}{dt} = 0.35 \frac{\Lambda}{q^2} \langle uv \rangle U' + v'q \tag{173}$$

(b) A $q - \Lambda$ model constructed from

$$\frac{dq}{dt} = - \frac{\langle uv \rangle}{q} U' - \frac{bq^2}{\Lambda} \tag{174}$$

$$\frac{d\Lambda}{dt} = 0.35 \frac{\Lambda}{q^2} \langle uv \rangle U' + v'q \tag{175}$$

together with the assumption that

$$\langle uv \rangle = -0.35 q \Lambda U' \tag{176}$$

An examination of Table 1 is instructive. First of all, the tensor scale model, with only two adjustable constants which were set from an experiment on grid turbulence, seems to do the best job overall. It gives by far the best prediction of asymptotic growth rate. The single scale model as normally used (which has the added parameter $c = 0.35$) is pretty good on all quantities except $q/\Lambda U'$ and the growth rate, where it is very poor indeed. The approximate $q - \Lambda$ which has still another adjustable parameter does least well for the parameters chosen, which are typical of these boundary-layer-like flows.

TABLE 1. COMPARISON OF THEORETICAL PREDICTIONS AND EXPERIMENTAL RESULTS FOR THE ASYMPTOTIC SOLUTION OF HOMOGENEOUS SHEAR FLOW

Parameter	Experiment (x/h = 11)	Theory					
		Full-Closure Tensor Scale	Diff. %	Full-Closure Tensor Scale	Diff. %	q - Λ Model	Diff. %
$Bv = -\frac{\langle uv \rangle}{q^2}$	0.149	0.1944	25%	0.1839	19%	0.221	33%
$Co = \frac{-\langle uv \rangle}{\sqrt{\langle uu \rangle \langle vv \rangle}}$	.47	.5547	16	.5207	10	NA	-
$q/\Lambda U'$	.67	.972	35	1.242	46	1.492	55
a	.076	.0729	4	.0287	165	.0345	120
$\langle uu \rangle /q^2$	.502	.5652	11	.5220	4	NA	-
$\langle vv \rangle /q^2$	.199	.2174	8	.2390	17	NA	-
$\langle ww \rangle /q^2$	.299	.2174	38	.2390	25	NA	-
$\frac{\langle vv \rangle + \langle ww \rangle}{q^2}$	.498	.4348	14	.4780	4	NA	-

A most interesting result is found if we make the assumption that, as indicated by the theoretical developments we have presented, the "constant" in the simple scale equation should be zero and not equal to 0.35 for homogeneous shear flows. If we do assume $c = 0$ in this formulation and find the asymptotic values of the basic parameters fcr homogeneous shear flow, we obtain almost the same values of the parameters that were obtained using the tensor scale equation. This is not surprising since the two formulations are now very similar. However, it must be remembered that it was the tensor scale equation which, when solved for homogeneous shear flow, showed that the production term should disappear from the scale equation.

If one puts $c = 0$ in the $q - \Lambda$ model we have concocted, the growth rate and the Bradshaw number become too large (0.963 and 0.257, respectively) while the parameter $q/\Lambda U'$ drops to 1.284 (which is not a great improvement).

3. CONCLUSIONS

In the previous sections, we have reviewed some of the characteristics of second-order modeling as it is currently used. One of the primary criticisms of these methods has been that they take no account of the structure that can be found in turbulent fields by modern instrumentation. We have given here an outline of how, by the use of a simple definition of tensor scale, second-order-closure models might be extended to take account of information on structure that can be gleaned from the two-point correlation equations. The tensor scale used is certainly not ideal for this purpose, but is was used not only because we are familiar with it but also because it illustrates many features that will be exhibited by any other definition of tensor scale.

We believe we have demonstrated two things in the results presented. First, we believe we have shown that there really cannot be such a thing as a universal scalar scale equation. Hence we believe that any steps taken to improve second-order-closure methods in the future must include, among other things, the derivation(from appropriate models of the two-point correlation tensor equations)multiple scales which will give a model that is compatible with the structure of the turbulent eddies that exist in a given mean flow. The method we have used here defines a tensor scale and uses a moment expansion to look at some general features of the structure problem that can be derived from a particular definition of tensor scale. The method is a good approximation for homogeneous flows. It is less justifiable for nonhomogeneous flows. However, we believe that, at the present time, it bears a relation to a more complete formulation, much like eddy viscosity methods bear to more complete formulations for calculating the Reynolds stress correlation $\overline{u_i'u_j'}$.

Second, we believe that we have shown that the homogeneous shear experiments are very powerful tools for the modeler. We believe that they do indeed have asymptotic solutions that are exponential and that when the asymptotic state is reached certain nondimensional parameters become constant. Since the asymptotic value of these parameters can be computed from a given model, the experimental results are an extremely useful tool for the development of valid models. One reason the measured values of these parameters are so useful is that they are independent of initial conditions and, in the past, arguments over initial conditions have been used to cover a multitude of modeling sins.

4. DISCUSSION

There is more than one way to look at any physical problem. In the field of turbulence research, it has become fashionable to define almost any unsteady flow in the wake of an unaccelerated body as turbulence. A lot of these unsteady flows contain large-scale, coherent structures and, thus, there has arisen a "large-scale eddy cult." The basic thesis of this group is that turbulence should or (for the extremists) must be attacked by some technique which identifies the turbulence with the interaction and decay of such large-scale structures. A corollary of this position is that closure methods do not or (for the extremists) cannot address themselves specifically to large-scale eddies and, therefore, are not really anything but dull, unphysical, and temporary methods for dealing with turbulent flows.

The authors of this paper do not believe that these people really understand the nature of closure calculations at the present time. Not only have closure methods demonstrated the existence of large-scale eddies in two cases (the roll eddies of the marine planetary boundary layer[11] and unsteady large eddies in the flow behind a rearward facing step[12]), but these eddies have been resolved in all their gory detail. This can be accomplished when the closure equations are used in their elliptic, time-dependent form and the grid spacing used is fine enough. Why is this so? It is because the Euler equations (which govern the formulation and a great deal of the behavior of large eddies) are contained in the time-dependent, elliptic equations. Thus we submit that closure techniques not only can describe large-scale eddies but must do so if the time-dependent, elliptic forms of the equations are used and the grid spacing used is small enough to resolve these eddies.

One further point. The Karman vortex wake is an unsteady flow associated with an unaccelerated body. We do not prefer to think of it as a form of structured turbulence (nor did those who first studied the phenomenon) although, if one were a member of "the cult," this point of view might be taken. The reason that we choose not to consider it as structured turbulence is that there is ample evidence of its existence in laminar flow. From the closure point of view, the Karman wake is looked at as an unsteady flow peculiar to the body that produces it. This unsteady flow interacts with itself to decay either through laminar exchange and dissipation or through turbulent exchange and dissipation, depending on the Reynolds number.

The research reported here is a description of our first attempts to make a closure theory of turbulence that is compatible with the large-scale structures that we must inevitably find when we run our closure codes in an elliptic manner. The work reported is one completed step in this direction. As noted in the text, we do not consider this work complete. We sincerely regret that we had to terminate this work before our attempt to construct a more general closure formulation could be completed. While we consider many of the detailed large-eddy studies to be outstanding, we also believe that elliptic, nonsteady, closure techniques will be the backbone of turbulence computations that will be of use to the military for the next twenty years, and that closure techniques that are compatible with this approach should and will be pursued.

REFERENCES

1. Batchelor, G. K., The Theory of Homogeneous Turbulence, The University Press, Cambridge (1960).

2. Rotta, J., Statistische Theorie nichthomogener Turbulenz, Z. Physik 129, 547 (1951).

3. Lighthill, M. J., On Sound Generated Aerodynamically, I. General Theory, Proc. Roy. Soc. (London) Ser. A, Vol. 211, pp. 564-587 (1952).

4. Taylor, G. I., Proc. London Math. Soc. 20, 196 (1921).

5. Rotta, J., Turbulente Stromungen, B. G. Teubner Stuttgart (1972).

6. Batchelor, G. K., Pressure Fluctuations in Isotropic Turbulence, Proc. Camb. Phil. Soc. 47, 359 (1951).

7. Donaldson, C. duP., Construction of a Dynamic Model of the Production of Atmospheric Turbulence and the Dispersal of Atmospheric Pollutants, Workshop on Micrometeorology (D. A. Haugen, ed.) American Meteorological Society, Boston (1973) pp. 313-392.

8. Donaldson, C. duP. and Sandri, G., On the Inclusion of Information on Eddy Structure in Second-Order-Closure Models of Turbulent Flows, presented and published in the Proceedings of the AGARD Symposium on the Fluid Dynamics of Jets with Applications to V/STOL, Lisbon, Portugal, AGARD-CP-308, November, 1981.

9. Donaldson, C. duP., The Relationship Between Eddy Transport and Second-Order-Closure Models for Stratified Media and for Vortices, in Free Turbulent Shear Flows, NASA SP-321, Vol. I, pp. 233-255 (1972).

10. Harris, V. G., Graham, J. A. H., and Corrsin, S., Further Experiments in Nearly Homogeneous Turbulent Shear Flow, J. Fluid Mechanics 81, pp. 657-687 (1977).

11. Lewellen, W. S., Teske, M. E., and Sheng, Y. P., Micrometeorological Applications of a Second-Order-Closure Model of Turbulent Transport, in Turbulent Shear Flows 2, Selected Papers for the Second International Symposium on Turbulent Shear Flows, Imperial College London, July 2-4, 1979, Springer-Verlag (New York), pp. 366-378.

12. Donaldson, C. duP, Lewellen, W. Stephen, Quinn, B., Sullivan, Roger D., Sykes, R. Ian, and Varma, Ashok K., A.R.A.P.'s Second-Order-Closure Model: Comparison with a Number of Complex Turbulent Flows, prepared for the 1980-81 AFOSR-HTTM-Stanford Conference on Complex Turbulent Flows: Comparison of Computation and Experiment, Stanford University, September 14-18, 1981 (Aeronautical Research Associates of Princeton, Inc., A.R.A.P. Report No. 469).

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